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V.A. Pletyukhov

ON THE CONJOINT DESCRIPTION MASSIVE AND MASSLESS SPIN 0 AND 1 FIELDS

In this work consideration is given to massless and massive gauge-invariant spin 0 and spin 1 fields (particles) within the scope of a theory of the generalized relativistic wave equations with an extended set of the Lorentz group representations. The results obtained may be useful as regards the application of a relativistic wave-equation theory in modern field models.

1. Introduction

One of the most extensively used ways to describe fundamental particles and fields is still a theory of relativistic wave equations (RWE), the foundations of which have been laid by Dirac [1], Fierz and Pauli [2; 3], Bhabha [4; 5], Harish-Chandra [6; 7], Gel'fand and Yaglom [8], Fedorov [9; 10]. This theory has been advanced proceeding from the assumption that a relativistic-invariant description of both massive and massless particles (fields) may always be reduced to a system of the first-order differential equations with constant factors, in the matrix form being given as follows:

$$\left(\gamma_\mu \partial_\mu + \gamma_0\right)\psi\left(x\right) = 0 \quad \left(\mu = 1 \div 4\right). \tag{1}$$

Here $\psi\left(x\right)$ is multicomponent wave function transformed in terms of some reducible Lorentz group representation T, γ_μ and γ_0 are square matrices.

In the case when the matrix γ_0 is nonsingular $\left(\det\gamma_0 \neq 0\right)$, equation (1) describing a massive particle may be reduced to the following form by multiplication into $m\gamma_0^{-1}$:

$$\left(\gamma_\mu \partial_\mu + mI\right)\psi\left(x\right) = 0, \tag{2}$$

where m is a parameter associated with mass, I is unity matrix.

A choice of the matrices γ_μ in equations (1) and (2) is limited by the following requirements (e.g., see [8; 9]):

i) invariance of the equation with respect to the transformations of its own Lorentz group;
ii) invariance with respect to reflections;
iii) possibility for derivation of the equation from the variational principle.

Equations of the form (2) meeting requirements i)–iii) are known as relativistic wave equations (RWE); equations of the form (1) with the same requirements are known as generalized RWE [9].

From requirement i) and from the condition of theory's irreducibility with respect to the Lorentz group it follows that the function ψ must be transformed by some set of linking irreducible Lorentz-group representations, forming what is known as a scheme for linking. The representations $\tau \sim \left(l_1, l_2\right)$ and $\tau' \sim \left(l'_1, l'_2\right)$ are referred to as linking if $l'_1 = l_1 \pm \frac{1}{2}, \quad l'_2 = l_2 \pm \frac{1}{2}$.

Aside from a choice of the wave function ψ, in definition of different spin and mass states possible for the particle described by equations (1) and (2) the matrices γ_4 and γ_0 are of particular importance. Properties of the matrix γ_4 are discussed comprehensively in [8].

A structure of the matrix γ_0 is determined in [5; 9]. Specifically, requirement i) results in reducibility of γ_0 to the diagonal form, the matrix being composed of independent scalar blocks corresponding to the irreducible representations of τ. For $\det\gamma_0 = 0$ some of these blocks are zero. As follows from requirement ii), nonzero elements a_τ of the matrix γ_0 satisfy the relation

$$a_\tau = a_{\dot\tau}, \quad (3)$$

where $\dot\tau$ is representation conjugate to τ with respect to the spatial reflection, i.e., if $\tau \sim (l_1, l_2)$, we have $\dot\tau \sim (l_2, l_1)$. In case of the finite-dimensional representations requirement iii) also leads to the relation of (3).

A distinctive feature of most well-known RWE of the form (2) (Dirac equation for spin $\frac{1}{2}$, Duffin-Kemmer equations for spins 0 and 1, Fierz-Pauli equation for spin $\frac{3}{2}$) is the fact that they involve a set of the Lorentz group representations *minimally necessary* for framing of a theory of this spin.

Such an approach in the case of $\det\gamma_0 = 0$ results in equations for zero-mass particles (e.g., Maxwell equations). Because of this, selection of $\det\gamma_0 = 0$ (also including $\gamma_0 = 0$) in a theory of RWE is associated with a description of massless particles [9; 11].

It is known that, as distinct from the description of massive particles, in a theory of massless particle with integer spin some of the wave-function components are unobservable (potentials) and others - observable (intensities). In consequence, for the potentials one can define the gauge transformations and impose additional requirements excluding «superfluous» components of ψ. But for the description of massive particles by RWE reducible to the form given by (2), the above-mentioned differentiation of the wave-function components is not the case. In other words, the notion of the gauge invariance of RWE (1) in the sense indicated previously is usually used for massless theories.

At the same time, there are papers, where so-called *massive gauge-invariant theories* are considered taking other approaches. Illustrative examples are furnished by Stückelberg's approach to the description of a massive spin 1 particle (see [12] and references herein) and by a $\hat{B} \wedge \hat{F}$-theory [13–16] claiming for the description of string interactions in 4-dimensional space and suggesting a mechanism (differing from Higgs's) of the mass generation due to gauge-invariant mixing of electromagnetic and massless vector fields with zero helicity. In the literature this field is called the Kalb-Ramond field [15;16] and the notoph [17]. Because of this, one should clear the question concerning the status of massive gauge-invariant fields in the theory of RWE.

Another feature of well-known RWE is the fact that on going from equation of the form (2) for a massive spin S particle to its massless analog of the form (1), by making the substitution $mI \to \gamma_0, \det\gamma_0 = 0$, not all of the helicity values from $+S$ to $-S$ are retained, a part of them is lost. This is the case when passing from the Duffin-Kemmer equation for spin 1 to Maxwell equations with the dropped-out zero helicity. In some modern models there is a necessity for simultaneous description of different massless fields [18]. Within the scope of a theory of RWE, it seems possible to solve this problem by the development of a scheme for passage from (2) to (1) RWE with the singular matrix γ_0 retaining not only maximal but also intermediate helicity values.

By authors' opinion, solution of the stated problems is important considering a possibility of using the well-developed apparatus of a theory based on RWE in modern theoretical field models including the phenomenological description of strings and superstrings in a space of the dimension $d=4$.

2. Gauge-invariant theories for massive spin 0 and 1 particles

Let us consider the following set of the Lorentz group irreducible representations in a space of the wave function ψ

$$(0,0)\oplus\left(\frac{1}{2},\frac{1}{2}\right)\oplus(0,1)\oplus(1,0). \tag{4}$$

The most general form of the corresponding (4) tensor system of the first-order equations meeting the requirements i) – iii) is given by

$$\alpha\partial_\mu\psi_\mu + a\psi_0 = 0, \tag{5}$$

$$\beta^*\partial_\nu\psi_{\mu\nu} + \alpha^*\partial_\mu\psi_0 + b\psi_\mu = 0, \tag{6}$$

$$\beta\left(-\partial_\mu\psi_\nu + \partial_\nu\psi_\mu\right) + c\psi_{\mu\nu} = 0. \tag{7}$$

Here ψ_0 is scalar, ψ_μ is vector, $\psi_{\mu\nu}$ is antisymmetric second-rank tensor; α,β are arbitrary dimensionless, generally speaking, complex parameters, and a,b,c are real nonnegative parameters, the dimension of which on selection of $\hbar=c=1$ is coincident with that of mass (massive parameters). Writing system (4) in the matrix form (1), we obtain in the basis

$$\psi = \left(\psi_0,\psi_\mu,\psi_{\mu\nu}\right) - column \tag{8}$$

for the matrix γ_0 the following expression:

$$\gamma_0 = \begin{pmatrix} a & & \\ & bI_4 & \\ & & cI_6 \end{pmatrix}. \tag{9}$$

(Matrices of the form γ_μ are not given as they are of no use for us in further consideration.)

In the general case, when none of the parameters in (4) is zero, this system describes a particle with a set of spins 0, 1 and with two masses

$$m_1 = \frac{\sqrt{ab}}{|\alpha|}, \quad m_2 = \frac{\sqrt{bc}}{|\beta|}, \tag{10}$$

the mass m_1 being associated with spin 0 and m_2 with spin 1. Omitting cumbersome calculations, we will verify this during analysis of special cases.

Imposing on the parameters of system (4) the requirement

$$\frac{\sqrt{a}}{|\alpha|} = \frac{\sqrt{c}}{|\beta|}, \tag{11}$$

we obtain RWE for a particle with spins 0, 1 and one mass $m=m_1=m_2$. At $\alpha=0$ system (4) goes to the Duffin-Kemmer equation of a particle with spin 1 and mass $m=m_2$

$$\beta^*\partial_\nu\psi_{\mu\nu} + b\psi_\mu = 0, \tag{12}$$

$$\beta\left(-\partial_\mu\psi_\nu + \partial_\nu\psi_\mu\right) + c\psi_{\mu\nu} = 0. \tag{13}$$

Finally, by setting in (4) $\beta = 0$, we arrive at the Duffin-Kemmer equation for a particle with spin 0 and mass $m = m_1$:

$$\alpha \partial_\mu \psi_\mu + a\psi_0 = 0, \tag{14}$$

$$\alpha^* \partial_\mu \psi_0 + b\psi_\mu = 0. \tag{15}$$

Now we consider the case that is of great interest for us, when the parameters a, b, c determining a structure of the matrix γ_0 in (9) are selectively set to zero.

In system (4) setting

$$a = 0, \tag{16}$$

we have the following system of equations:

$$\partial_\mu \psi_\mu = 0, \tag{17}$$

$$\beta^* \partial_\nu \psi_{\mu\nu} + \alpha^* \partial_\mu \psi_0 + b\psi_\mu = 0, \tag{18}$$

$$\beta\left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu\right) + c\psi_{\mu\nu} = 0, \tag{19}$$

that, being written in the matrix form of (1), corresponds in basis (8) to the singular matrix γ_0

$$\gamma_0 = \begin{pmatrix} 0 & & \\ & bI_4 & \\ & & cI_6 \end{pmatrix}. \tag{20}$$

From system (16) one can easily derive the second-order equations

$$\Box \psi_0 = 0 \tag{21}$$

$$\Box \psi_\mu - \frac{c\alpha^*}{|\beta|^2} \partial_\mu \psi_0 - \frac{bc}{|\beta|^2} \psi_\mu = 0. \tag{22}$$

As regards the scalar function ψ_0 governed by equation (21), the following aspects must be taken into account. System (16) is invariant with respect to the gauge transformations

$$\psi_0 \to \psi_0 - \frac{1}{\alpha^*}\Lambda, \quad \psi_\mu \to \psi_\mu + \frac{1}{b}\partial_\mu \Lambda, \tag{23}$$

where the gauge function Λ is limited by the constraint

$$\Box \Lambda = 0. \tag{24}$$

From comparison between (24) and (21) it follows that the function ψ_0 acts as a gauge function and hence provides no description for a physical field. In other words, gauge transformations (23) and (24) make it possible to impose an additional condition

$$\psi_0 = 0. \tag{25}$$

In this case system (16) is transformed to system (11) describing a massive spin 1 particle, whereas equation (22), considered simultaneously with (17), goes to an ordinary Proca equation. In this way the gauge invariance of system (16), as compared to (4), leads to a decrease in physical degrees of freedom from four to three, exclusive of the spin 0 state.

Note that a similar result may be obtained without the explicit use of the considerations associated with the gauge invariance. By the introduction of

$$\phi_\mu = \psi_\mu + \frac{\alpha^*}{b}\partial_\mu\psi_0 \tag{26}$$

system (16) may be directly reduced to the form

$$\beta^*\partial_\nu\psi_{\mu\nu} + b\phi_\mu = 0, \tag{27}$$

$$\beta\left(-\partial_\mu\phi_\nu + \partial_\nu\phi_\mu\right) + c\psi_{\mu\nu} = 0 \tag{28}$$

coincident with (11).

This variant of a gauge-invariant theory is known [12] as a Stueckelberg approach to the description of a massive spin 1 particle. We have considered this variant for a complete study of the possibilities given by system (4).

In (4) we set

$$c = 0. \tag{29}$$

Then the initial system of equations (4) takes the form

$$\alpha\partial_\mu\psi_\mu + a\psi_0 = 0, \tag{30}$$

$$\beta^*\partial_\nu\psi_{\mu\nu} + \alpha^*\partial_\mu\psi_0 + b\psi_\mu = 0, \tag{31}$$

$$-\partial_\mu\psi_\nu + \partial_\nu\psi_\mu = 0. \tag{32}$$

According to (10), it should describe a particle with the mass $m_1 = \frac{\sqrt{ab}}{|\alpha|}$ and with spin 0. By convolution of equation (31) with the operator ∂_μ we have

$$\Box\psi_0 + \frac{b}{\alpha^*}\partial_\mu\psi_\mu = 0. \tag{33}$$

Comparing (33) with (30), we arrive at the equation

$$\Box\psi_0 - \frac{ab}{|\alpha|^2}\psi_0 = 0, \tag{34}$$

that provides support for all the afore-said.

The states associated with spin 1, for the condition set by (29), disappear due to the invariance of system (29) with respect to the gauge transformations

$$\psi_{\mu\nu} \to \psi_{\mu\nu} - \frac{1}{\beta^*}\Lambda_{\mu\nu}, \quad \psi_\mu \to \psi_\mu + \frac{1}{b}\partial_\nu\Lambda_{\mu\nu}, \tag{35}$$

where an arbitrary choice of the gauge function $\Lambda_{\mu\nu}$ is constrained by

$$\partial_\alpha\partial_\nu\Lambda_{\mu\nu} - \partial_\mu\partial_\nu\Lambda_{\alpha\nu} = 0. \tag{36}$$

On the other hand, as follows from equations (31), (32), a similar equation

$$\partial_\alpha\partial_\nu\psi_{\mu\nu} - \partial_\mu\partial_\nu\psi_{\alpha\nu} = 0 \tag{37}$$

is satisfied by the tensor $\psi_{\mu\nu}$. Consequently, a choice of $\Lambda_{\mu\nu}$ is arbitrary enough to impose an additional constraint

$$\partial_\nu\psi_{\mu\nu} = 0 \tag{38}$$

that is in accord with (37). In this case system (29) takes the form of (14), i.e. it actually describes a massive spin 0 particle.

Note also that system (29) may be reduced to the form

$$\alpha\partial_\mu\phi_\mu + a\psi_0 = 0, \tag{39}$$

$$\alpha^* \partial_\mu \psi_0 + b\phi_\mu = 0 \tag{40}$$

similar to (14) by introduction of the vector

$$\phi_\mu = \psi_\mu + \frac{\beta^*}{b} \partial_\nu \psi_{\mu\nu}. \tag{41}$$

Thus, the considered variant of a massive gauge-invariant theory is some kind of an analog for the Stückelberg approach but applicable to the description of a spin 0 particle. The authors have not found any mentioning of such a description in the literature available.

In the formalism of RWE (1) this theory is consistent with the matrix γ_0 of the form

$$\gamma_0 = \begin{pmatrix} a & & \\ & bI_4 & \\ & & 0_6 \end{pmatrix}. \tag{42}$$

Next we consider a set of the Lorentz group representations

$$\left(\frac{1}{2},\frac{1}{2}\right) \oplus \left(\frac{1}{2},\frac{1}{2}\right)' \oplus (0,1) \oplus (1,0), \tag{43}$$

where the representation $\left(\frac{1}{2},\frac{1}{2}\right)'$ conforms to the pseudovector or to the absolutely antisymmetric third-rank tensor. The most general form of a tensor system of the first-order equations based on representation (43) and meeting the above-mentioned requirements i)–iii) is given by

$$\alpha \partial_\nu \psi_{\mu\nu} + a\psi_\mu = 0, \tag{44}$$

$$\beta \partial_\nu \tilde{\psi}_{\mu\nu} + b\tilde{\psi}_\mu = 0, \tag{45}$$

$$\alpha^* \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu\right) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta + c\psi_{\mu\nu} = 0. \tag{46}$$

Here $\tilde{\psi}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\psi_{\alpha\beta}$, $\tilde{\psi}_\mu = \frac{1}{6}\varepsilon_{\mu\nu\alpha\beta}\psi_{\nu\alpha\beta}$, $\varepsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor ($\varepsilon_{1234} = -i$), $\psi_{\nu\alpha\beta}$ is antisymmetric third-rank tensor, α, β are still arbitrary dimensionless, generally speaking, complex parameters, a, b, c are mass parameters.

Writing system (43) in the form (1), where $\Psi = \left(\psi_\mu, \tilde{\psi}_\mu, \psi_{\mu\nu}\right)$ is column, for the matrix γ_0 we get the expression

$$\gamma_0 = \begin{pmatrix} aI_4 & & \\ & bI_4 & \\ & & cI_6 \end{pmatrix}. \tag{47}$$

Now we elaborate on massive gauge-invariant theories obtainable from (43) by manipulations with the parameters a, b, c.

Let us take the case

$$a = 0. \tag{48}$$

In this case we have a system of equations

$$\partial_\nu \psi_{\mu\nu} = 0, \tag{49}$$
$$\beta \partial_\nu \tilde{\psi}_{\mu\nu} + b\tilde{\psi}_\mu = 0, \tag{50}$$
$$\alpha^* \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu \right) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta + c\psi_{\mu\nu} = 0 \tag{51}$$

that, when formulated as (1), is associated with the singular matrix γ_0

$$\gamma_0 = \begin{pmatrix} 0_4 & & \\ & bI_4 & \\ & & cI_6 \end{pmatrix}. \tag{52}$$

From (48) we can obtain the second-order equations

$$(\Box - \frac{bc}{|\beta|^2})\tilde{\psi}_\mu = 0, \tag{53}$$
$$\partial_\mu \tilde{\psi}_\mu = 0, \tag{54}$$
$$\Box \psi_\mu - \partial_\mu \partial_\nu \psi_\nu = 0. \tag{55}$$

Equations (53) and (54) denote that system (48) involves the description of a massive spin 1 particle. As shown by equation (55), system (48) describes also a massless field with the potential ψ_μ. The latter allows for involvement of the gauge transformation

$$\psi_\mu \to \psi_\mu + \partial_\mu \Lambda \tag{56}$$

(Λ is arbitrary function), with respect to which system (48) and equation (55) are invariant. The indicated invariance means that this massless field is a Maxwell-type field with helicity ± 1.

In this manner the gauge-invariant system (48) irreducible with respect to the Lorentz group offers a simultaneous description of a massive spin 1 particle and of a massless field with helicity ± 1. In other words, here we deal with a massive-massless gauge-invariant theory rather than massive theory, as is the case for (16) and (29).

A similar result may be obtained if we set in (48)

$$b = 0. \tag{57}$$

Then we have

$$\gamma_0 = \begin{pmatrix} aI_4 & & \\ & 0_4 & \\ & & cI_6 \end{pmatrix}, \tag{58}$$

and the second-order equations following from the corresponding first-order system

$$\alpha \partial_\nu \psi_{\mu\nu} + a\psi_\mu = 0, \tag{59}$$
$$\partial_\nu \tilde{\psi}_{\mu\nu} = 0, \tag{60}$$
$$\alpha^* \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu \right) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta + c\psi_{\mu\nu} = 0 \tag{61}$$

are of the form

$$(\Box - \frac{ac}{|\alpha|^2})\psi_\mu = 0, \tag{62}$$

$$\partial_\mu \psi_\mu = 0, \tag{63}$$

$$\Box \tilde{\psi}_\mu - \partial_\mu \partial_\nu \tilde{\psi}_\nu = 0. \tag{64}$$

Equation (64) and system (58) are invariant with respect to the gauge transformations

$$\tilde{\psi}_\mu \to \tilde{\psi}_\mu + \partial_\mu \tilde{\Lambda}. \tag{65}$$

Thus, here we deal again with a gauge-invariant massive-massless spin 1 theory.

Let us consider another set of representations

$$(0,0)' \oplus \left(\frac{1}{2},\frac{1}{2}\right)' \oplus (0,1) \oplus (1,0) \quad , \tag{66}$$

where $(0,0)'$ is associated with the absolutely antisymmetric fourth-rank tensor $\psi_{\mu\nu\alpha\beta}$. The most general tensor formulation of RWE based on the set of representations given in (66) takes the form

$$\alpha \partial_{[\mu} \psi_{\nu\alpha\beta\,]} + a\psi_{\mu\nu\alpha\beta} = 0, \tag{67}$$

$$\alpha^* \partial_{[\nu} \psi_{\alpha\beta\,]} + \beta^* \partial_\mu \psi_{\mu\nu\alpha\beta} + b\psi_{\nu\alpha\beta} = 0, \tag{68}$$

$$\beta \partial_\nu \psi_{\nu\alpha\beta} + c\psi_{\alpha\beta} = 0, \tag{69}$$

where the following notation is used:

$$\partial_{[\nu} \psi_{\alpha\beta\,]} \equiv \partial_\nu \psi_{\alpha\beta} + \partial_\beta \psi_{\nu\alpha} + \partial_\alpha \psi_{\beta\nu}, \tag{70}$$

$$\partial_{[\mu} \psi_{\nu\alpha\beta\,]} \equiv \partial_\mu \psi_{\nu\alpha\beta} - \partial_\nu \psi_{\mu\alpha\beta} + \partial_\alpha \psi_{\mu\nu\beta} - \partial_\beta \psi_{\mu\nu\alpha}. \tag{71}$$

After introduction into system (66) of the dual conjugates $\tilde{\psi}_{\mu\nu}, \tilde{\psi}_\mu$ and pseudoscalar $\tilde{\psi}_0 = \frac{1}{4!}\varepsilon_{\mu\nu\alpha\beta}\psi_{\mu\nu\alpha\beta}$ instead of the tensors $\psi_{\mu\nu}, \psi_{\nu\alpha\beta}, \psi_{\mu\nu\alpha\beta}$, it is conveniently rewritten to give

$$\alpha \partial_\mu \tilde{\psi}_\mu + a\tilde{\psi}_0 = 0, \tag{72}$$

$$\beta^* \partial_\nu \tilde{\psi}_{\mu\nu} + \alpha^* \partial_\mu \tilde{\psi}_0 + b\tilde{\psi}_\mu = 0, \tag{73}$$

$$\beta\left(-\partial_\mu \tilde{\psi}_\nu + \partial_\nu \tilde{\psi}_\mu\right) + c\tilde{\psi}_{\mu\nu} = 0. \tag{74}$$

As seen from the comparison between (72) and (4), these systems are dual in that one may be derived from the other by the substitutions

$$\psi_0 \leftrightarrow \tilde{\psi}_0, \;\; \psi_\mu \leftrightarrow \tilde{\psi}_\mu, \;\; \psi_{\mu\nu} \leftrightarrow \tilde{\psi}_{\mu\nu}. \tag{75}$$

Clearly, the use of system (72) with the aim of framing various gauge-invariant theories on its basis follows the same procedure and gives the same results as with system (4). So, when in (72) we set $a = 0$, a gauge-invariant theory for a pseudoscalar particle of the mass $\frac{\sqrt{bc}}{|\beta|}$ is put forward. But setting $c = 0$, we arrive at a gauge-invariant theory for a pseudoscalar particle of the mass $\frac{\sqrt{ab}}{|\alpha|}$.

3. Simultaneous description of massless fields

Returning to a set of representations (4) and to tensor system (4), we consider the case

$$b = 0. \tag{76}$$

The following system is obtained:

$$\alpha \partial_\mu \psi_\mu + a\psi_0 = 0, \tag{77}$$

$$\beta^* \partial_\nu \psi_{\mu\nu} + \alpha^* \partial_\mu \psi_0 = 0, \tag{78}$$

$$\beta\left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu\right) + c\psi_{\mu\nu} = 0 \tag{79}$$

that in basis (8) is associated with the matrix γ_0 of the form

$$\gamma_0 = \begin{pmatrix} a & & \\ & 0_4 & \\ & & cI_4 \end{pmatrix}. \tag{80}$$

From system (76) we get d'Alembert equation (21) for the scalar function ψ_0 and the second-order equation

$$\psi_\mu - \left(1 - \frac{c|\alpha|^2}{a|\beta|^2}\right) \partial_\mu \partial_\nu \psi_\nu = 0 \tag{81}$$

for the vector ψ_μ. From this it is inferred that we deal with a massless field. When considering the quantities ψ_0 and ψ_μ as potentials of this field, we treat equation (79) as a definition of the intensity $\psi_{\mu\nu}$ in terms of the potentials, (77) is additional constraint similar to the Feynman gauge. Then equation (78) acts as an equation of motion.

With this treatment, system (76) and equation (81) is invariant with respect to the gauge transformation

$$\psi_\mu \to \psi_\mu + \partial_\mu \Lambda, \tag{82}$$

where an arbitrary choice of Λ is constrained by (24). Gauge transformations (82) and (24) in combination with an additional requirement (77) indicate that, among the four components of the potential ψ_μ, only two components are independent. They describe a transverse component of the field under study. One more, longitudinal, component of this field is described by the scalar function ψ_0. In this way a choice of (76) in system (4) leads to a theory of a massless filed with three helicity values $\pm 1, 0$. This is one of the distinguishing features of system (4) as opposed to a theory of Duffin–Kemmer for spin 1, that on a similar passage to the limit results in a massless field with helicities ± 1.

Also, note that equation (81) with due regard for (77) may be rewritten as

$$\Box \psi_\mu + \left(1 - \frac{c|\alpha|^2}{a|\beta|^2}\right) \frac{a}{\alpha} \partial_\mu \psi_0 = 0, \tag{83}$$

from whence it follows that a gradient of the scalar component acts as an (internal) source of the transverse component of this massless field.

Next we select the case when in system (4)

$$a = 0, \quad b = 0. \tag{84}$$

The resultant system

$$\partial_\mu \psi_\mu = 0, \tag{85}$$

$$\beta^* \partial_\nu \psi_{\mu\nu} + \alpha^* \partial_\mu \psi_0 = 0, \tag{86}$$

$$\beta \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu \right) + c \psi_{\mu\nu} = 0 \tag{87}$$

is distinguished from system (76) by the potential gauge requirement (compare (77) with (85)). In this case the matrix γ_0 is of the form

$$\gamma_0 = \begin{pmatrix} 0 & & \\ & 0_4 & \\ & & cI_6 \end{pmatrix}. \tag{88}$$

From (84) one can obtain equation (21) for the function ψ_0 and the second-order equation

$$\Box \psi_\mu - \frac{\alpha^* c}{|\beta|^2} \partial_\mu \psi_0 = 0 \tag{89}$$

for ψ_μ that, similar to system (84), is invariant with respect to gauge transformations (82), (24). All this indicates that we deal again with two interrelated massless fields: vector field with helicity ± 1 and scalar field with helicity 0, the gradient of a scalar field acting as a source of the vector field.

The other two massless analogs of system (4), when

$$a = 0, \quad c = 0 \tag{90}$$

and

$$b = 0, \quad c = 0, \tag{91}$$

are associated with the description of a massless field of zero helicity. Establishing this fact, we will not concern ourselves with the details.

Considering the possibility for simultaneous description of different massless fields, we next analyze a set of the representations in (43) and the first-order system of (43).

First, we take the case

$$c = 0, \quad a = b. \tag{92}$$

In this case system (43) is of the form

$$\alpha \partial_\nu \psi_{\mu\nu} + a \psi_\mu = 0, \tag{93}$$

$$\beta \partial_\nu \tilde{\psi}_{\mu\nu} + a \tilde{\psi}_\mu = 0, \tag{94}$$

$$\alpha^* \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu \right) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta = 0, \tag{95}$$

and the matrix γ_0 (47) is transformed to the matrix

$$\gamma_0 = \begin{pmatrix} aI_8 & \\ & O_6 \end{pmatrix}. \tag{96}$$

In (92) we take components of the tensor $\psi_{\mu\nu}$ as potentials, assuming the vector ψ_μ and the pseudovector $\tilde{\psi}_\mu$ as intensities. Then equations (93) and (94) are the intensity definitions in terms of the potentials and (95) acts as an equation of motion.

From system (92) we derive the second-order equation for the tensor-potential $\psi_{\mu\nu}$

$$\Box \psi_{\mu\nu} = 0. \tag{97}$$

Equations (92) and (97) are invariant with respect to the gauge transformations

$$\psi_{\mu\nu} \to \psi_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad (98)$$

where an arbitrary choice of the functions Λ_μ is constrained by

$$\Box \Lambda_\mu - \partial_\mu \partial_\nu \Lambda_\nu = 0. \quad (99)$$

Equation (97) and gauge transformations (98) and (99) indicate that a choice of (92) leads to a theory for a massless particle of zero helicity carrying spin 1 in the process of interactions.

By the present time, two approaches to the description of such a particle have been known: (1) Ogievetsky and Polubarinov approach [17] in which an intensity is represented by the vector (in [17] this particle is called the notoph) and (2) Kalb-Ramond approach [13], where an intensity is represented by the antisymmetric third-rank tensor or pseudovector (Kalb-Ramond field). System (92) combines the description of both fields in one irreducible RWE.

In a sense this pattern may be complemented if in (43) we set

$$a = 0, \quad b = 0. \quad (100)$$

As a result, we have the following system:

$$\partial_\nu \psi_{\mu\nu} = 0, \quad (101)$$

$$\partial_\nu \tilde{\psi}_{\mu\nu} = 0, \quad (102)$$

$$\alpha^* \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu\right) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta + c\psi_{\mu\nu} = 0 \quad (103)$$

that is associated with the matrix γ_0 of the form

$$\gamma_0 = \begin{pmatrix} O_8 & \\ & cI_6 \end{pmatrix}. \quad (104)$$

In system (100) the components ψ_μ and $\tilde{\psi}_\mu$ are naturally considered as potentials, and $\psi_{\mu\nu}$ is taken as an intensity. Then it is invariant with the gauge transformations

$$\psi_\mu \to \psi_\mu + \Lambda_\mu, \tilde{\psi}_\mu \to \tilde{\psi}_\mu + \tilde{\Lambda}_\mu, \quad (105)$$

where an arbitrary choice of the gauge functions $\Lambda_\mu, \tilde{\Lambda}_\mu$ is constrained by

$$\alpha^* \left(-\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu\right) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\Lambda}_\beta = 0. \quad (106)$$

In other words, at $\alpha = \beta = 1$ system (100) represents the well-known two-potential formulation from electrodynamics (e.g., see [19]) for a massless spin 1 field with helicity ± 1.
Thus, a reciprocal complementarity of the theories based on systems (92) and (100) is exhibited in their mathematical structure, including that of the matrix γ_0, and also in interpretations of the field components ψ_μ, $\tilde{\psi}_\mu$, $\psi_{\mu\nu}$ as well as in properties (helicity) of the particles described.

Of particular interest is the case when in (43) we set

$$a = 0, \quad c = 0. \quad (107)$$

This results in the system

$$\partial_\nu \psi_{\mu\nu} = 0, \quad (108)$$

$$\beta \partial_\nu \tilde{\psi}_{\mu\nu} + b\tilde{\psi}_\mu = 0, \quad (109)$$

$$\alpha^* \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu\right) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta = 0 \quad (110)$$

and leads to the matrix

$$\gamma_0 = \begin{pmatrix} O_4 & & \\ & bI_4 & \\ & & O_6 \end{pmatrix}. \tag{111}$$

For convenience, we rewrite (107) in the following form:

$$\partial_\nu \psi_{\mu\nu} = 0, \tag{112}$$

$$\beta\left(\partial_\mu \psi_{\nu\alpha} + \partial_\alpha \psi_{\mu\nu} + \partial_\nu \psi_{\alpha\mu}\right) + b\psi_{\mu\nu\alpha} = 0, \tag{113}$$

$$\alpha^*\left(-\partial_\nu \psi_\alpha + \partial_\alpha \psi_\nu\right) + \beta^* \partial_\mu \psi_{\mu\nu\alpha} = 0, \tag{114}$$

where $\psi_{\mu\nu\alpha}$ is antisymmetric third-rank tensor dual with respect to the pseudovector $\tilde{\psi}_\mu$.

According to the structure of system (111), ψ_μ and $\psi_{\mu\nu}$ are potentials, $\psi_{\mu\nu\alpha}$ is intensity. Then equation (113) is a definition of the intensity, and (112) acts as an additional constraint imposed on the tensor-potential $\psi_{\mu\nu}$ and included originally in the system itself. This constraint leaves for tensor $\psi_{\mu\nu}$ satisfying the second-order equation

$$\Box \psi_{\mu\nu} + \frac{|\alpha|^2}{|\beta|^2} \frac{b}{\alpha}\left(\partial_\mu \psi_\nu - \partial_\nu \psi_\mu\right) = 0 \tag{115}$$

two independent components. As this takes place, system (111) is invariant with respect to relative gauge transformations (98), (99). Due to an arbitrary choice of the gauge function Λ_μ constraining by condition (99) we have only one independent component for $\psi_{\mu\nu}$ that is associated with the state of a massless field with zero helicity.

To elucidate a meaning of the term $\partial_\mu \psi_\nu - \partial_\nu \psi_\mu$ in (115), we turn to the potential ψ_μ. Apart from transformations (98), (99), system (111) is also invariant with respect to the gauge transformation

$$\psi_\mu \to \psi_\mu + \partial_\mu \Lambda, \tag{116}$$

where Λ is arbitrary function . From equation (114) for ψ_μ we derive the second-order equation

$$\Box \psi_\mu - \partial_\mu \partial_\nu \psi_\nu = 0, \tag{117}$$

in combination with (99) indicating that the potential ψ_μ gives description for the transverse component (helicity ± 1) of the massless field under study. The expression

$$\partial_\mu \psi_\nu - \partial_\nu \psi_\mu \equiv F_{\mu\nu} \tag{118}$$

in equations (114) and (115) may be considered as an intensity associated with this transverse component. Then equation (114) rewritten with regard to the notation of (118) as

$$\beta^* \partial_\mu \psi_{\mu\nu\alpha} - \alpha^* F_{\nu\alpha} = 0, \tag{119}$$

acts as an equation of motion in system (111).

Thus, a choice (107) of mass parameters in the initial system (43) leads to a theory of the generalized massless field with polarizations 0, $\pm$ 1.

Selection of the parameters

$$b = 0, \quad c = 0. \tag{120}$$

in system (43) also results in a theory of the generalized massless field with helicities 0, $\pm$ 1 featuring a dual conjugate of that obtainable in the case of (107). Details are beyond the scope of this paper.

4. Mass generation and rwe theory

In 1974 in the works [13; 14] a mechanism of mass generation was proposed differing from the well-known Higgs mechanism. Later this mechanism has been identified as a gauge-invariant field mixing. It's essence is as follows. Two massless systems of equations are considered cooperatively as initial systems

$$\partial_\nu \psi_{\mu\nu} = 0, \tag{121}$$

$$-\partial_\mu \phi_\nu + \partial_\nu \phi_\mu + \psi_{\mu\nu} = 0, \tag{122}$$

and

$$\partial_\mu \psi_{\mu\nu\alpha} = 0, \tag{123}$$

$$-\partial_\mu \phi_{\nu\rho} - \partial_\nu \phi_{\rho\mu} - \partial_\rho \phi_{\mu\nu} + \psi_{\mu\nu\rho} = 0, \tag{124}$$

the first system describing an electromagnetic field and the second one describing field of Kalb-Ramond. In (123) and (124) tensor $\psi_{\mu\nu\alpha}$ is considered to be an intensity. Then into the Lagrangian of this system an additional term is included

$$L_{int} = m\phi_\mu \partial_\nu \phi_{\mu\nu} \tag{125}$$

without violation of the gauge-invariance for the initial Lagrangian L_0. This term may be formally treated as an interaction of the fields under study (so-called topological interaction). Varying the Lagrangian $L = L_0 + L_{int}$ and introducing the pseudovector $\tilde{\psi}_\mu = \frac{1}{3!}\varepsilon_{\mu\nu\alpha\beta}\psi_{\nu\alpha\beta}$, we have a system

$$\partial_\nu \psi_{\mu\nu} + m\tilde{\psi}_\mu = 0, \tag{126}$$

$$-\partial_\mu \tilde{\psi}_\nu + \partial_\nu \tilde{\psi}_\mu + m\psi_{\mu\nu} = 0, \tag{127}$$

$$\partial_\nu \tilde{\phi}_{\mu\nu} + \tilde{\psi}_\mu = 0, \tag{128}$$

$$-\partial_\mu \phi_\nu + \partial_\nu \phi_\mu + \psi_{\mu\nu} = 0, \tag{129}$$

where

$$\tilde{\phi}_{\mu\nu} = \frac{1}{2!}\varepsilon_{\mu\nu\alpha\beta}\phi_{\alpha\beta}. \tag{130}$$

Now in system (125) we replace ϕ_μ and $\tilde{\phi}_{\mu\nu}$ by the quantities $\tilde{G}_\mu$ and $G_{\mu\nu}$ using the formulae

$$\tilde{G}_\mu = \phi_\mu - \frac{1}{m}\tilde{\psi}_\mu, \tag{131}$$

$$G_{\mu\nu} = \tilde{\phi}_{\mu\nu} - \frac{1}{m}\psi_{\mu\nu}. \tag{132}$$

Finally, system (125) is reduced to the following form:

$$\partial_\nu \psi_{\mu\nu} + m\tilde{\psi}_\mu = 0, \tag{133}$$

$$-\partial_\mu \tilde{\psi}_\nu + \partial_\nu \tilde{\psi}_\mu + m\psi_{\mu\nu} = 0, \tag{134}$$

$$\partial_\nu G_{\mu\nu} = 0, \tag{135}$$

$$-\partial_\mu \tilde{G}_\nu + \partial_\nu \tilde{G}_\mu = 0. \tag{136}$$

As seen, system (133) is reducible with respect to the Lorentz group into subsystems (133), (134) and (135), (136). The first of them describing a massive spin 1 particle is interpreted in [13] as an interaction transporter between open strings. Subsystem (135), (136) gives no description for a physical field, as it is associated with zero energy density. However, its presence is necessary to impart to the latter the status of a gauge-invariant theory.

Using the formalism of generalized RWE, all the above may be interpreted as follows. Let us consider a set of representations

$$\left(\frac{1}{2},\frac{1}{2}\right)\oplus\left(\frac{1}{2},\frac{1}{2}\right)'\oplus 2(1,0)\oplus 2(0,1), \tag{137}$$

associated with tensor system (4), (123).

It is obvious that on the basis of (137) one can derive RWE (1) with the matrices

$$\gamma_\mu=\begin{pmatrix}\gamma_\mu^{DK} & \\ & \gamma_\mu^{DK}\end{pmatrix},\quad \gamma_0=\begin{pmatrix}O_4 & & & \\ & I_6 & & \\ & & O_6 & \\ & & & I_4\end{pmatrix}, \tag{138}$$

where γ_μ^{DK} are 10-dimensional Duffin-Kemmer matrices. Introduction into the Lagrangian of a topological term (125) results in the changed form of the matrices γ_μ leaving the matrix γ_0 unaltered. Substitutions of (130) are equivalent to the unitary transformation restoring the form of γ_μ matrix given in (138). As this is the case, the matrix γ_0 takes the form

$$\gamma_0=\begin{pmatrix}mI_{10} & \\ & O_{10}\end{pmatrix}. \tag{139}$$

In this way we actually arrive at RWE reducible to the ordinary Duffin-Kemmer equation for a massive spin 1 particle and at the massless fermionic limit of this equation. Nontrivial nature of the mass generation method, from the viewpoint of a theory of RWE, consists in the fact that on passage from the initial massless field(s) to the massive one neither the form of γ_μ matrices nor the rank of singular γ_0 matrix is affected, the procedure being reduced to permutation of zero and unity blocks of this matrix only. In the process the number of degrees of freedom (that is equal to three) for a field system is invariable; it seems as if the notoph passes its degree of freedom to the photon, that automatically leads to a massive spin 1 particle.

5. Discussion and conclusions

Based on the examples considered, the following important conclusions can be drawn.

<u>Conclusion 1.</u> *Generalized RWE (1) with the singular matrix* γ_0 *can describe not only massless but also massive fields (particles). Featuring the gauge invariance, these equations just form the class of massive gauge-invariant theories.*

As demonstrated in Sec. II using equations (48) and (58) as an example, a theory of generalized RWE suggests also a variant of the generalized description for massive and massless fields based on RWE irreducible with respect to the Lorentz group. Thus, we arrive at the following conclusion.

<u>Conclusion 2.</u> *RWE of the form (1) with the singular matrix* γ_0 *can describe the*

fields involving both massive and massless components. In this case it is more correct to refer to massive-massless gauge-invariant theories rather than to the massive ones.

As demonstrated in Sec. III, within the scope of RWE (1), on adequate selection of the Lorentz group representations in a space of the wave function ψ and interpretation of its components, one can give the description of a massless field not only with helicity ± 1 but also with helicity 0 as well as simultaneous description of the indicated fields. Generalizing this result for the case of arbitrary spin S, we can make the following conclusion.

Conclusion 3. *A theory of the generalized RWE with the singular matrix γ_0 makes it possible to describe not only massless fields with maximal (for the given set of representations) helicity $\pm S$, but also fields with intermediate helicity values as well as to offer a simultaneous description of these fields.*

It is clear that, all other things being equal, a character of the field described by equation (1) with the singular matrix γ_0 is dependent on the form of this matrix. To find when the singular matrix γ_0 leads to massless theories and when it results in massive or massive-massless gauge-invariant theories, we examine the Lorentz structure of the «massive» term $\gamma_0\psi$ in the foregoing cases. It is observed that in the case of (8), (16), (20) associated with a massive gauge-invariant spin 1 theory the matrix γ_0 (20) affecting the wave function ψ (8) in the expression $\gamma_0\psi$ retains (without reducing to zero) the Lorentz covariants $\psi_\mu, \psi_{\mu\nu}$, on the basis of which an ordinary (of the form (2)) massive spin 1 theory can be framed. But in the case of a massless theory given by (8), (76), (80) the matrix γ_0 in the expression $\gamma_0\psi$ retains the covariant $\psi_{\mu\nu}$, on the basis of which it is impossible to frame RWE of the form (2) for a massive particle. A similar pattern is characteristic for the remaining cases: in all the massive (massive-massless) gauge-invariant theories the matrix γ_0 affecting the wave function ψ retains its covariant components necessary for framing of an ordinary massive spin 1 or 0 theory; provided the expression $\gamma_0\psi$ doesn't involve such a necessary set of covariants, massless theories can be framed only. This leads us to the fourth conclusion.

Conclusion 4. *Should the generalized RWE (1) with the singular matrix* γ_0 in the product $\gamma_0\psi$ *retain a set of the Lorentz covariants sufficient to frame an ordinary (with det $\gamma_0 \neq 0$) theory of a massive spin S particle, this RWE may be associated with a massive gauge-invariant spin S theory. Otherwise, when this requirement is not fulfilled for any S, RWE (1) can describe a massless field only.*

Proceeding from all the afore-said, we arrive at the following important though obvious conclusion.

Conclusion 5. *To frame both massive (massive-massless) gauge-invariant spin S theory and massless theory with intermediate helicity values from $+S$ to $-S$ we need an extended, in comparison with a minimally necessary for the description of this spin (helicity), set of the irreducible Lorentz group representations in a space of the wave function ψ.*

In the present work, when considering spin 1, the above-mentioned extension has been accomplished by the introduction of scalar representation $(0,0)$ into a set of the representations given by (4) and of pseudoscalar representation $\left(\frac{1}{2},\frac{1}{2}\right)'$ – into a set given by (43). Greater potentialities are offered by the use of the multiple (recurrent) Lorentz group representations.

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В.А. Плетюхов **О совместном описании массивных и безмассовых полей со спинами 0 и 1**

Рассматриваются безмассовые и массивные калибровочно-инвариантные поля со спинами 1 и 0 с точки зрения теории обобщенных релятивистских волновых уравнений. Получены новые уравнения, которые могут быть использованы в современных теоретико-полевых моделях.