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Finite groups with restrictions on normal subgroups

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Abstract. We investigate the structure of finite soluble groups all whose proper normal subgroups belong to some class of groups, namely a Fitting class and the class of all supersoluble groups.

1. Introduction

All groups in this paper are finite.

Let $\mathfrak{F}$ be a class of groups. We say that a group G is a minimal non-$\mathfrak{F}$-group or a $\mathfrak{F}$-critical group if $G \notin \mathfrak{F}$ but all proper subgroups of G belong to $\mathfrak{F}$. G. Miller and H. Moreno [12] studied minimal non-abelian groups in 1903. Minimal non-nilpotent groups were first investigated by O. Schmidt [18]. Such groups are also called Schmidt groups, and their properties are well known [14], [17]. B. Huppert [7, Theorem 22], K. Doerk [4] and V. Nagrebeckij [16] studied minimal non-supersoluble groups, see also [2].

It is natural to study groups in which only some proper subgroups belong to a class $\mathfrak{F}$, for instance, normal subgroups or subgroups of prime index.

S. Levischenko [10] studied groups whose subgroups of non-prime index are nilpotent. V. Monakhov [13] described the structure of groups whose subgroups of non-prime index are supersoluble. G. Malanjina and G. Shevcov [11] investigated the structure of a soluble group such that its commutator subgroup is nilpotent and all proper normal subgroups are supersoluble. L. Kazarin and Y. Korzukov [9] obtained the description of a soluble group with trivial Frattini subgroup and a supersoluble normal subgroup of prime index.

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In this paper we investigate soluble groups all whose proper normal subgroups belong to a Fitting class. We also obtain new properties of a soluble group all whose proper normal subgroups are supersoluble.

2. Preliminaries

By $A \le B$ we denote that A is a subgroup of a group B. If $A \le B$ and $A \ne B$, then A is called a proper subgroup of B and denoted by $A < B$. Let A and B be subgroups of a group G, and $G = AB$. If $G \ne AB_1$ for every proper subgroup B_1 of B, then B is called a minimal supplement to A in G. By G_p we denote a Sylow p-subgroup of a group G; $G_{p'}$ denotes a Hall p'-subgroup of G. We also use $\pi(G)$ to denote the set of all prime devisors of $|G|$.

Let $\pi_{ind}(G)$ be the set of all primes p such that in G there exists a normal subgroup of index p. If G is a soluble group, then $\pi_{ind}(G) \ne \emptyset$.

By $\mathfrak{A}$, $\mathfrak{N}$ and $\mathfrak{U}$ we denote the class of all abelian, nilpotent and supersoluble groups, respectively. We use the term "$\mathfrak{F}$-group" to denote a group belonging to a class of groups $\mathfrak{F}$. A class of groups $\mathfrak{F}$ is said to be s-closed if every subgroup of G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$. A class of groups $\mathfrak{F}$ is said to be s_n-closed if every subnormal subgroup of G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$.

A class of groups $\mathfrak{F}$ is a Fitting class if $\mathfrak{F}$ has the following two properties:

(i) if $G \in \mathfrak{F}$ and H is a normal subgroup of G, then $H \in \mathfrak{F}$;

(ii) if N_1 and N_2 are normal subgroups of G and $N_1, N_2 \in \mathfrak{F}$, then $N_1N_2 \in \mathfrak{F}$.

If $\mathfrak{F}$ is a Fitting class, then the subgroup $G_{\mathfrak{F}} = \langle N \lhd G : N \in \mathfrak{F}\rangle$ is the largest normal $\mathfrak{F}$-subgroup of G, and it is called the $\mathfrak{F}$-radical of G.

A formation is a class of groups $\mathfrak{F}$ with the following two properties:

(i) if $G \in \mathfrak{F}$ and $N \lhd G$, then $G/N \in \mathfrak{F}$;

(ii) if N_1 and N_2 are normal subgroups of G and $G/N_1, G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

If $\mathfrak{F}$ is a formation, then the subgroup $G^{\mathfrak{F}} = \bigcap\{N \lhd G : G/N \in \mathfrak{F}\}$ is the smallest normal subgroup of G with quotient in $\mathfrak{F}$, and it is called the $\mathfrak{F}$-residual of G. A Fitting class which is also a formation is said to be a Fitting formation.

If $\mathfrak{X}$ and $\mathfrak{F}$ are classes of groups, we define their class product $\mathfrak{X}\mathfrak{F}$ as $\mathfrak{X}\mathfrak{F} = \{G \in \mathfrak{E} : N \lhd G, N \in \mathfrak{X} \Rightarrow G/N \in \mathfrak{F}\}$. Here $\mathfrak{E}$ denotes the class of all groups. Let $\mathfrak{X}$ be a class of group and $\mathfrak{F}$ a formation. We define $\mathfrak{X} \circ \mathfrak{F} = \{G \in \mathfrak{E} : G^{\mathfrak{F}} \in \mathfrak{X}\}$ and call $\mathfrak{X} \circ \mathfrak{F}$ the formation product of $\mathfrak{X}$ with $\mathfrak{F}$. Clearly, if $\mathfrak{X}$ is an s_n-closed class of group, then $\mathfrak{X}\mathfrak{F} = \mathfrak{X} \circ \mathfrak{F}$. Therefore, if $\mathfrak{X}$ and $\mathfrak{F}$ are formations, then $\mathfrak{X}\mathfrak{F} = \mathfrak{X} \circ \mathfrak{F}$ [5, p. 337].

All unexplained notations and terminology are standard. The reader is referred to [5], [8] if necessary.

We need the following results.

Lemma 1 (see e.g. [3], [5]). *Let $\mathfrak{X}$ and $\mathfrak{F}$ be formations, let G be a group, $H \le G$ and $K \lhd G$. Then*

(1) *$G \in \mathfrak{F}$ if and only if $G^{\mathfrak{F}} = 1$;*

(2) *if $\mathfrak{X} \subseteq \mathfrak{F}$, then $G^{\mathfrak{F}} \le G^{\mathfrak{X}}$; in particular, $G^{\mathfrak{U}} \le G^{\mathfrak{N}} \le G^{\mathfrak{A}} = G'$;*

(3) *$(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$;*

(4) *$\mathfrak{X}\mathfrak{F}$ is a formation;*

(5) *$G^{(\mathfrak{X}\mathfrak{F})} = (G^{\mathfrak{F}})^{\mathfrak{X}}$.*

Lemma 2. *Let $\mathfrak{F}$ be an s_n-closed class and G a soluble group. Then the following statements are equivalent:*

(1) *every proper normal subgroup of G belongs to $\mathfrak{F}$;*

(2) *every normal subgroup of G of prime index belongs to $\mathfrak{F}$;*

(3) *the commutator subgroup G' and every proper subgroup of G containing G' belong to $\mathfrak{F}$.*

Proof. Obviously, (1) implies (2) and (3).

Now we check that (1) follows from (2). Let N be a proper normal subgroup of a soluble group G. In G there exists a maximal normal subgroup M such that $N \subseteq M$ and $|G : M| = p$ for some $p \in \pi(G)$. Then by the choice of G, $M \in \mathfrak{F}$ and $N \in \mathfrak{F}$ since $\mathfrak{F}$ is an s_n-closed class.

Finally we prove that (1) follows from (3). Let N be a proper normal subgroup of a soluble group G. There exists a maximal normal subgroup M of G such that $N \subseteq M$ and $|G : M| = p$ for some $p \in \pi(G)$. The quotient group G/M is abelian and $G' \le M$. Hence by the hypothesis, $M \in \mathfrak{F}$ and $N \in \mathfrak{F}$ since $\mathfrak{F}$ is an s_n-closed class. □

Lemma 3. *Let $\mathfrak{F}$ be an s_n-closed class and G a soluble group. Every proper normal subgroup of G belongs to $\mathfrak{F}$ if and only if $G = G'H$, where H is a minimal supplement to G' in G, and $G'K \in \mathfrak{F}$ for every proper subgroup K of H.*

Proof. Let $\mathfrak{F}$ be an s_n-closed class and G a soluble group. Suppose that every proper normal subgroup of G belongs to $\mathfrak{F}$, and let H be a minimal supplement to G' in G. If K is a proper subgroup of H, then $G'K$ is a proper normal subgroup of G. Therefore $G'K \in \mathfrak{F}$.

Conversely, suppose $G = G'H$, where H is a minimal supplement to G' in G, and $G'K \in \mathfrak{F}$ for every proper subgroup K of H. If X is a proper subgroup of G such that $G' \subseteq X$, then $X = G'(X \cap H)$ and $X \in \mathfrak{F}$ since $X \cap H < H$. By Lemma 2, every proper normal subgroup of G belongs to $\mathfrak{F}$. □

Lemma 4. *If every two elements of a group G generate a supersoluble subgroup, then G is supersoluble* [8, VI.9.18].

Lemma 5. *Let A and B be normal supersoluble subgroups of G and let $G = AB$.*

(1) *If the commutator subgroup G' is nilpotent, then G is supersoluble* [1].

(2) *If the indices $|G : A|$ and $|G : B|$ are relatively prime, then G is supersolvable* [6].

3. Soluble groups whose normal subgroups belong to a Fitting class

Theorem 1. *Let $\mathfrak{F}$ be a Fitting class and G a soluble non-$\mathfrak{F}$-group. Every proper normal subgroup of G belongs to $\mathfrak{F}$ if and only if the following two conditions hold:*

(1) *$\pi_{ind}(G) = \{p\}$ for some $p \in \pi(G)$ and $G = G'\langle x\rangle$, where $\langle x\rangle$ is a minimal supplement to G' in G and $x \in G_p$;*

(2) *$G' = G^{\mathfrak{N}}$, $G_{\mathfrak{F}} = G'\langle x^p\rangle$ and $|G : G_{\mathfrak{F}}| = p$.*

Proof. Let G be a soluble non-$\mathfrak{F}$-group such that every proper normal subgroup of G belongs to a Fitting class $\mathfrak{F}$. By [5, Lemma II.2.10(a)], G has a unique maximal normal subgroup M. Therefore all proper normal subgroups of G are contained in M, $|G : M| = p$ and $\pi_{ind}(G) = \{p\}$. In particular, $G_{\mathfrak{F}} = M$ and $G_p^G = G$. If $G^{\mathfrak{N}}G_p \neq G$, then $G^{\mathfrak{N}}G_p$ is a proper normal subgroup of G, and $G_p^G = G \leq G^{\mathfrak{N}}G_p$, a contradiction. Hence $G^{\mathfrak{N}}G_p = G$ and $G/G^{\mathfrak{N}}$ is a p-group. Since $G/G^{\mathfrak{N}}$ has a unique maximal normal subgroup, $G/G^{\mathfrak{N}}$ is a cyclic p-group. Let X be a subgroup of minimal order in G_p such that $G = G^{\mathfrak{N}}X$. Then $G^{\mathfrak{N}} \cap X \leq \Phi(X)$ [15, 3.21]. Since $G/G^{\mathfrak{N}} \simeq X/G^{\mathfrak{N}} \cap X$ is a cyclic p-group, it follows that $X = \langle x\rangle$ is also a cyclic p-group [15, 3.20]. Consequently, $G = G^{\mathfrak{N}}\langle x\rangle$, where $x \in G_p$. Since $G^{\mathfrak{N}}\langle x^p\rangle$ is a proper normal subgroup of G of prime index p, we have $G^{\mathfrak{N}}\langle x^p\rangle = M = G_{\mathfrak{F}}$.

Note $G^{\mathfrak{N}} \leq G'$. On the other hand, $G' \leq G^{\mathfrak{N}}$ since $G/G^{\mathfrak{N}}$ is a cyclic p-group. Thus, $G^{\mathfrak{N}} = G'$.

Conversely, let $\mathfrak{F}$ be a Fitting class, G a soluble non-$\mathfrak{F}$-group satisfying (1) and (2). Then $G_{\mathfrak{F}}$ is the unique maximal normal subgroup of G and so every proper normal subgroup of G belongs to $\mathfrak{F}$. $\square$

Let r be a prime. A group G is r-closed if its Sylow r-subgroup is normal. The class of all r-closed groups is an s-closed Fitting formation, and it coincides with the product $\mathfrak{N}_r\mathfrak{E}_{r'}$. Here $\mathfrak{N}_r$ is the class of all r-groups, $\mathfrak{E}_{r'}$ is the class of all groups whose order is prime to r.

Taking $\mathfrak{F} = \mathfrak{N}_r\mathfrak{E}_{r'}$ in Theorem 1, we obtain

Corollary 1. *Let r be a prime and G a non-r-closed soluble group. Every proper normal subgroup of G is r-closed if and only if $G = G'\langle x\rangle$, where $x \in G_r$, $\langle x\rangle$ is a minimal supplement to G' in G, $G'\langle x^r\rangle$ is r-closed, and $|G : G'\langle x^r\rangle| = r$.*

Proof. Let r be a prime and G a non-r-closed soluble group such that every proper normal subgroup of G is r-closed. Then every proper normal subgroup of G belongs to $\mathfrak{N}_r\mathfrak{E}_{r'}$. Since $\mathfrak{N}_r\mathfrak{E}_{r'}$ is a Fitting formation, we can use Theorem 1. By Theorem 1(1), $G = G'\langle x\rangle$, where $x \in G_p$ for some $p \in \pi(G)$. If $p \neq r$, then $|G : G'|$ is prime to r and G is r-closed since G' is r-closed, contrary to the choice of G. Hence $p = r$. In view of Theorem 1(2), $G'\langle x^r\rangle$ is r-closed and $|G : G'\langle x^r\rangle| = r$.

Conversely, let r be a prime and G a non-r-closed soluble group such that $G = G'\langle x\rangle$, where $x \in G_r$, and $G'\langle x^r\rangle$ is r-closed. Then by Lemma 3, every proper normal subgroup of G is r-closed. $\square$

A group G is r-nilpotent if there exists a Hall r'-subgroup $G_{r'}$ which is normal in G. The class of all r-nilpotent groups is an s-closed Fitting formation, and it coincides with the product $\mathfrak{E}_{r'}\mathfrak{N}_r$.

Taking $\mathfrak{F} = \mathfrak{E}_{r'}\mathfrak{N}_r$ in Theorem 1, we obtain

Corollary 2. *Let r be a prime and G a non-r-nilpotent soluble group. Every proper normal subgroup of G is r-nilpotent if and only if $G = G'\langle x\rangle$, where $x \in G_p$ for some $p \in \pi(G) \setminus \{r\}$, $\langle x\rangle$ is a minimal supplement to G' in G, and $G'\langle x^p\rangle$ is r-nilpotent.*

Proof. Since $\mathfrak{E}_{r'}\mathfrak{N}_r$ is a Fitting formation, we can use Theorem 1. By Theorem 1 (1) $G = G'\langle x\rangle$, where $x \in G_p$ for some $p \in \pi(G)$. If $p = r$, then $G_{r'} \leq G'$. Since G' r-nilpotent, we obtain $G_{r'}$ is normal in G and G is r-nilpotent. This contradicts to the choice of G. Therefore $p \neq r$. The conclusion follows from Theorem 1. $\square$

Example 1. A_4, E_4 and 1 are the proper normal subgroups of the symmetric group S_4 of order 4. Therefore all proper normal subgroups of S_4 are 2-closed and 3-nilpotent. At the same time the commutator subgroup $(S_4)' = A_4$ is not a Hall subgroup.

A group G is r-decomposable if it is r-closed and r-nilpotent. A class of all r-decomposable groups is an s-closed Fitting formation, and it coincides with the intersection $\mathfrak{N}_r\mathfrak{E}_{r'} \cap \mathfrak{E}_{r'}\mathfrak{N}_r$.

Taking $\mathfrak{F} = \mathfrak{N}_r\mathfrak{E}_{r'} \cap \mathfrak{E}_{r'}\mathfrak{N}_r$ in Theorem 1, we obtain

Corollary 3. *Let r be a prime and G a non-r-decomposable soluble group. Every proper normal subgroup of G is r-decomposable if and only if $G = G'\langle x\rangle$, where $x \in G_p$ for some $p \in \pi(G)$, $\langle x\rangle$ is a minimal supplement to G' in G, $G'\langle x^p\rangle$ is r-decomposable, and either G is r-closed and $p \neq r$ or G is r-nilpotent and $p = r$.*

Proof. Since every proper normal subgroup of G is r-closed and r-nilpotent, we can use Corollary 1 and 2. Hence $G = G'\langle x\rangle$, where $x \in G_p$ for some $p \in \pi(G)$, and $G'\langle x^p\rangle$ is r-decomposable. If G is r-nilpotent, then it is non-r-closed and by Corollary 1 $p = r$. If G is non-r-nilpotent, then, by Corollary 2 $p \neq r$ and by Corollary 1, G is r-closed. □

Taking $\mathfrak{F} = \mathfrak{N}$ in Theorem 1, we obtain

Corollary 4. *Let G be a soluble non-nilpotent group. Every proper normal subgroup of G is nilpotent if and only if the following two conditions hold:*

(1) *$\pi_{ind}(G) = \{p\}$ for some $p \in \pi(G)$ and $G = [G']\langle x\rangle$, where $\langle x\rangle$ is a Sylow p-subgroup of G;*

(2) *$G' = G^{\mathfrak{N}}$ and $F(G) = [G']\langle x^p\rangle$.*

Proof. Let G be a soluble non-nilpotent group all whose proper normal subgroup are nilpotent. When $\mathfrak{F} = \mathfrak{N}$, we can use Theorem 1. Since a Hall p'-subgroup $G_{p'}$ of G is contained in $G^{\mathfrak{N}}$ and $G^{\mathfrak{N}}$ is nilpotent, it implies $G_{p'}$ is normal in G. Consequently, the quotient group $G/G_{p'}$ is a p-group and $G^{\mathfrak{N}} = G_{p'}$. Hence $G = [G^{\mathfrak{N}}]\langle x\rangle$ and $\langle x\rangle$ is a Sylow p-subgroup of G. The conclusion follows from Theorem 1. □

4. Soluble groups with supersoluble normal subgroups

It is well known that the class $\mathfrak{U}$ is not a Fitting class. Therefore, we can not use Theorem 1 for $\mathfrak{F} = \mathfrak{U}$.

Lemma 6. *If every proper normal subgroup of a soluble non-supersoluble group G is supersoluble, then the following statements hold:*

(1) *$G/G^{\mathfrak{N}}$ is a p-group for some $p \in \pi(G)$; in particular, $\pi_{ind}(G) = \{p\}$;*

(2) *G/G' is cyclic or a direct product of two cyclic p-groups;*

(3) *if the quotient group G/G' is non-cyclic, then*
   (3.1) *G' is non-nilpotent;*
   (3.2) *$G^{\mathfrak{U}} = (G')^{\mathfrak{N}} \leq G''$;*
   (3.3) *$G^{\mathfrak{N}} \leq F(G) < G'$; in particular, $G/G^{\mathfrak{N}}$ is non-abelian.*

Proof. (1) Let G be a soluble non-supersoluble group all whose proper normal subgroups are supersoluble. Suppose that $G/G^{\mathfrak{N}}$ is not primary. If $A/G^{\mathfrak{N}}$ is a Sylow p-subgroup and $B/G^{\mathfrak{N}}$ is a Hall p'-subgroup of $G/G^{\mathfrak{N}}$, then A and B are normal proper subgroups of G. Therefore, A and B are supersoluble. Thus $G = AB$ and G is supersoluble by Lemma 5(2). This contradiction implies that $G/G^{\mathfrak{N}}$ is a p-group for some $p \in \pi(G)$. In particular, $\pi_{ind}(G) = \{p\}$.

(2) Since $G^{\mathfrak{N}} \leq G'$, G/G' is a abelian p-group. Therefore, G/G' can be decomposed as a direct product of cyclic p-subgroups:

$$G/G' = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots \times \langle a_t \rangle. \tag{1}$$

If $\langle x, y \rangle G'$ is supersoluble for all $x, y \in G$, then G is also supersoluble by Lemma 4. This contradiction implies that the subgroup $\langle x, y \rangle G'$ is not supersoluble for some $x, y \in G$. Since $\langle x, y \rangle G'$ is normal in G, it follows that $\langle x, y \rangle G'$ coincides with G and G/G' has at most two generators. Thus $t \leq 2$ in (1).

(3) Suppose that G/G' is not cyclic. Let A/G' and B/G' be different subgroups of index p. Then $G = AB$, subgroups A and B are normal in G and supersoluble. In view of Lemma 5(1), G' is non-nilpotent. This implies that $(G')^{\mathfrak{N}} \neq 1$ by Lemma 1(1).

By Lemma 1(5) $G^{\mathfrak{N}\mathfrak{A}} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}}$. As $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$, it follows that $G^{\mathfrak{N}\mathfrak{A}} = (G')^{\mathfrak{N}} \leq G^{\mathfrak{U}}$. Since

$$G/(G')^{\mathfrak{N}} = A/(G')^{\mathfrak{N}} \cdot B/(G')^{\mathfrak{N}}, \ (G/(G')^{\mathfrak{N}})' = G'/(G')^{\mathfrak{N}},$$

$G/(G')^{\mathfrak{N}}$ is supersoluble by Lemma 5(1). Therefore, $G^{\mathfrak{U}} \leq (G')^{\mathfrak{N}} \leq G''$. Thus, $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} \leq G''$.

The subgroups A' and B' are normal in G and nilpotent, it implies that $A'B' \leq F(G)$. The subgroups $A(A'B')/(A'B')$ and $B(A'B')/(A'B')$ of the quotient group $G/(A'B')$ are abelian and normal, so $G/(A'B')$ is nilpotent and $G^{\mathfrak{N}} \leq A'B' < G'$. Since $G^{\mathfrak{N}} \neq G'$, $G/G^{\mathfrak{N}}$ is non-abelian. □

Statements (2) and (3.1) of Lemma 6 were first obtained in [11, Theorem 1]. We give the more concise proof.

Theorem 2. *If every proper normal subgroup of a soluble non-supersoluble group G is supersoluble, then the following statements hold:*

(1) *if the quotient group $G/G^{\mathfrak{N}}$ is non-cyclic, then*
  (1.1) $G = [G^{\mathfrak{N}}]G_p$ *for some* $p \in \pi(G)$*;*
  (1.2) $F(G) = G^{\mathfrak{N}} \times \mathrm{O}_p(G)$*;*
  (1.3) *all proper subgroups of* $G_p/\mathrm{O}_p(G)$ *are abelian;*
  (1.4) *the subgroup* $[G^{\mathfrak{N}}]P$ *is supersoluble for all* $P < G_p$*;*

(2) *if the quotient group $G/G^{\mathfrak{N}}$ is cyclic, then*
  (2.1) $\pi_{ind}(G) = \{p\}$ *for some* $p \in \pi(G)$ *and* $G = G^{\mathfrak{N}}\langle x\rangle$*, where* $\langle x\rangle$ *is a minimal supplement to* $G^{\mathfrak{N}}$ *in* G *and* $x \in G_p$*;*
  (2.2) $G^{\mathfrak{N}} = G'$ *and* $G^{\mathfrak{N}}\langle x^p\rangle$ *is supersoluble.*

Conversely, if Statements (1.1) *and* (1.4) *or* (2.1) *and* (2.2) *hold for G, then every proper normal subgroup of G is supersoluble.*

Proof. Let G be a soluble non-supersoluble group all whose proper normal subgroups are supersoluble. In view of Lemma 6(1), the quotient group $G/G^{\mathfrak{N}}$ is a p-group for some $p \in \pi(G)$. Since $G^{\mathfrak{N}} \leq G'$, G/G' is an abelian p-group and $G = G'G_p$.

Case 1. The quotient group $G/G^{\mathfrak{N}}$ is non-cyclic.

Let $A/G^{\mathfrak{N}}$ and $B/G^{\mathfrak{N}}$ be different subgroups of index p. Hence $G = AB$ and the subgroups A and B are normal in G and supersoluble. If G' is nilpotent, then G is supersoluble by Lemma 5(1). This contradiction implies that G' is not nilpotent. Since the subgroups A' and B' are normal in G and nilpotent, it follows that $A'B' \leq F(G)$. The subgroups $A(A'B')/(A'B')$ and $B(A'B')/(A'B')$ are abelian and normal in $G/(A'B')$. Then $G/(A'B')$ is nilpotent and $G^{\mathfrak{N}} \leq A'B'$. Therefore $G^{\mathfrak{N}}$ is a nilpotent Hall p'-subgroup of G and $F(G) = G^{\mathfrak{N}} \times \mathrm{O}_p(G)$. Since G' is not nilpotent, $G^{\mathfrak{N}} < G'$ and G_p is non-abelian.

Let X be a maximal subgroup of G_p such that $\mathrm{O}_p(G) \subseteq X$. Hence $H = [G^{\mathfrak{N}}]X$ is normal in G and supersoluble. This implies that H' is normal in G and nilpotent, therefore $H' \leq F(G) = G^{\mathfrak{N}} \times \mathrm{O}_p(G)$. Consequently, $X/\mathrm{O}_p(G)$ is abelian and all proper subgroups of $G_p/\mathrm{O}_p(G)$ are abelian.

Let $P < G_p$ and P_1 be a maximal subgroup of G_p such that $\mathrm{O}_p(G) \subseteq P_1$. Then $[G^{\mathfrak{N}}]P_1$ is normal in G and supersoluble. Hence $[G^{\mathfrak{N}}]P$ is supersoluble.

Case 2. The quotient group $G/G^{\mathfrak{N}}$ is cyclic.

Since $G/G^{\mathfrak{N}}$ is cyclic and $G^{\mathfrak{N}} \leq G'$, we obtain $G^{\mathfrak{N}} = G'$. Suppose that X is a subgroup of the minimal order in G_p such that $G = G'X$. Then $G' \cap X \leq \Phi(X)$ [15, 3.21]. Hence $G/G' \simeq X/G' \cap X$ is a cyclic p-group. It follows that $X = \langle x \rangle$ is also a cyclic p-group [15, 3.20]. Consequently, $G = G'\langle x \rangle$, where $x \in G_p$, and $G'\langle x^p \rangle$ is a supersoluble normal subgroup in G of index p.

Conversely, let G be a a soluble non-supersoluble group satisfying (1.1) and (1.4) or (2.1) and (2.2). In view of Lemma 3, every proper normal subgroup of G is supersoluble. □

Corollary 5. *If G is a non-supersoluble group such that the commutator subgroup G' is nilpotent and all proper normal subgroups of G are supersoluble, then*

(1) *$G = [G']\langle x \rangle$, where $\langle x \rangle$ is a Sylow p-subgroup of G for some $p \in \pi(G)$;*

(2) *$[G']\langle x^p \rangle$ is supersoluble.*

Conversely, if G is a group such that the commutator subgroup G' is nilpotent and Statements (1)–(2) *hold, then all proper normal subgroups of G are supersoluble* [11, Theorem 2].

Proof. Since G' is nilpotent and $G^{\mathfrak{N}} \leq G'$, $G^{\mathfrak{N}}$ is a Hall p'-subgroup of G by Lemma 6(1). In view of Lemma 6(3.1), the quotient group $G/G^{\mathfrak{N}}$ is cyclic. The conclusion follows from Theorem 2(2). □

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