On grouops with formational subnormal or self-normalizing subgroups

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Abstract

We establish the structure of finite groups with $\mathfrak{F}$-subnormal or self-normalizing primary cyclic subgroups in case $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups.

Keywords: finite group, primary cyclic subgroup, derived subgroup, residual, subnormal subgroup, abnormal subgroup.

1 Introduction

All groups in this paper are finite. We use the standard notation and terminology of [1–3].

Let $\mathfrak{F}$ be a formation, and let G be a group. A subgroup H is called $\mathfrak{F}$-subnormal if either $G = H$ or there is a chain of subgroups

$$H = H_0 < \cdot \ H_1 < \cdot \ \ldots < \cdot \ H_n = G$$

such that $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all i, this is equivalent to $H_i^{\mathfrak{F}} \leq H_{i-1}$. Here $A_B = \bigcap_{b\in B} A^b$ is the core of a subgroup A in a group B, $H_{i-1} < \cdot \ H_i$ denotes that H_{i-1} is a maximal subgroup of a group H_i. A subgroup H of a group G is said to be $\mathfrak{F}$-abnormal in G if $L/K_L \notin \mathfrak{F}$ for all subgroups K and L such that $H \leq K < \cdot \ L \leq G$. It is clear that any proper subgroup of a group can not be both $\mathfrak{F}$-subnormal and $\mathfrak{F}$-abnormal, i. e. these notions are alternative. Besides, if $\mathfrak{X} \subseteq \mathfrak{F}$, then every $\mathfrak{X}$-subnormal subgroup is $\mathfrak{F}$-subnormal and every $\mathfrak{F}$-abnormal subgroup is $\mathfrak{X}$-abnormal.

Many authors investigated groups in which all or certain subgroups are $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal, see references in [4].

For a subgroup-closed formation $\mathfrak{F}$ containing all nilpotent groups, every $\mathfrak{F}$-abnormal subgroup is self-normalizing. Self-normalizingness and $\mathfrak{F}$-subnormality are not alternative notions. For instance, in a soluble group, every non-normal subgroup of prime index is both self-normalizing and $\mathfrak{U}$-subnormal. Here $\mathfrak{U}$ denotes the formation of all supersoluble groups.

Example. Assume that $\mathfrak{F} = \mathfrak{N}\mathfrak{A}$ is the formation of all groups with the nilpotent derived subgroups. The class of groups with $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal primary subgroups was investigated in [5]. If we replace $\mathfrak{F}$-abnormality by self-normalizingness, then the class under study broadens.

By E_{p^n} we denote an elementary abelian group of order p^n for a prime p and a positive integer n, Z_m denotes a cyclic group of order m for a positive integer m.

In GAP's SmallGroup library [6], there is the group

$$G = (S_3 \times S_3 \times A_4) \rtimes Z_2 \quad (\texttt{GAP}\ \textit{SmallGroup ID}\ [864, 4670]).$$

In G, the Sylow 3-subgroup $G_3 \simeq E_{3^3}$ is $\mathfrak{F}$-subnormal, the Sylow 2-subgroup $G_2 \simeq E_{2^4} \rtimes Z_2$ is self-normalizing, non-$\mathfrak{F}$-subnormal and non-$\mathfrak{F}$-abnormal, and every proper subgroup of G_2 is $\mathfrak{F}$-subnormal. Besides,

$$G^{\mathfrak{F}} = F(G) \simeq E_{3^2} \times E_{2^2} < G^{\mathfrak{N}} \simeq E_{3^2} \times A_4 < G' \simeq (E_{3^2} \times A_4) \rtimes Z_2.$$

Thus G belongs to the class of groups with $\mathfrak{F}$-subnormal or self-normalizing primary subgroups and does not belong to the class of groups in which primary subgroups are $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal.

Groups in which certain subgroups are $\mathfrak{F}$-subnormal or self-normalizing were studied in [7]– [9]. In particular, in [9] the structure of group with $\mathfrak{F}$-subnormal or self-normalizing Sylow subgroups was described for the large class of subgroup-closed formations $\mathfrak{F}$.

We proceed to develop this line of research and describe groups with $\mathfrak{F}$-subnormal or self-normalizing primary cyclic subgroups in case $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups. We prove

Theorem. *If $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups, then for a soluble group $G \notin \mathfrak{F}$, the following statements are equivalent.*

(1) *Every primary cyclic subgroup of G is self-normalizing or $\mathfrak{F}$-subnormal.*

(2) *Every proper subgroup of G is self-normalizing or $\mathfrak{F}$-subnormal.*

(3) *$G = G' \rtimes \langle x \rangle$, where $\langle x \rangle$ is a Sylow p-subgroup for some $p \in \pi(G)$ and a Carter subgroup, $G' \rtimes \langle x^p \rangle \in \mathfrak{F}$.*

A subnormal subgroup-closed formation $\mathfrak{F}$ is superradical if a group $G = AB$, where A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of G, belongs to $\mathfrak{F}$. It is well known that a formation with the Shemetkov property [10, 6.4.6] and a lattice formation [11, Lemma 4] are superradical.

2 Preliminaries

If A is a subgroup of a group B, then we write $A \leq B$; if A is a normal subgroup of a group B, then we write $A \lhd B$. By $\pi(G)$ we denote the set of all primes dividing the order of a group G. A semidirect product of a normal subgroup A and a subgroup B is denoted by $A \rtimes B$. The symbol $\square$ indicates the end of the proof.

The formations of all abelian and nilpotent subgroups are denoted by $\mathfrak{A}$ and $\mathfrak{N}$, respectively.

Let $\mathfrak{F}$ be a formation, and G be a group. The subgroup

$$G^{\mathfrak{F}} = \bigcap \{N \lhd G : G/N \in \mathfrak{F}\}$$

is called the $\mathfrak{F}$-residual of G.

If $\mathfrak{X}$ and $\mathfrak{F}$ are subgroup-closed formations, then the product

$$\mathfrak{X}\mathfrak{F} = \{\ G \in \mathfrak{E} \mid G^{\mathfrak{F}} \in \mathfrak{X}\}$$

is also a subgroup-closed formation according to [2, p. 337] and [3, p. 191].

We need the following properties of $\mathfrak{F}$-subnormal and $\mathfrak{F}$-abnormal subgroups.

Lemma 1. *Let $\mathfrak{F}$ be a formation, let H and K be subgroups of G, and let $N \lhd G$. The following statements hold.*

(1) *If K is $\mathfrak{F}$-subnormal in H and H is $\mathfrak{F}$-subnormal in G, then K is $\mathfrak{F}$-subnormal in G* [10, 6.1.6 (1)].

(2) *If K/N is $\mathfrak{F}$-subnormal in G/N, then K is $\mathfrak{F}$-subnormal in G* [10, 6.1.6 (2)].

(3) *If H is $\mathfrak{F}$-subnormal in G, then HN/N is $\mathfrak{F}$-subnormal in G/N* [10, 6.1.6 (3)].

(4) *If $\mathfrak{F}$ is a subgroup-closed formation and $G^{\mathfrak{F}} \leq H$, then H is $\mathfrak{F}$-subnormal in G* [10, 6.1.7 (1)].

(5) *If $\mathfrak{F}$ is a subgroup-closed formation, $K \leq H$, H is $\mathfrak{F}$-subnormal in G and $H \in \mathfrak{F}$, then K is $\mathfrak{F}$-subnormal in G.*

Proof. (5) Since $\mathfrak{F}$ is a subgroup-closed formation and $H \in \mathfrak{F}$, we have K is $\mathfrak{F}$-subnormal in H and K is $\mathfrak{F}$-subnormal in G in view of (1). $\square$

Lemma 2 ([7, Lemma 1.4]). *Let $\mathfrak{F}$ be a subgroup-closed formation containing groups of order p for all $p \in \mathbb{P}$, and let A be a $\mathfrak{F}$-abnormal subgroup of G.*

(1) *If $A \leq B \leq G$, then A is $\mathfrak{F}$-abnormal in B and $A = N_G(A)$;*

(2) *If $A \leq B \leq G$, then B is $\mathfrak{F}$-abnormal in G and $B = N_G(B)$.*

A subgroup H of a group G is called an $\mathfrak{X}$-projector of G if HN/N is an $\mathfrak{X}$-maximal subgroup of G/N for every normal subgroup N of G. A Carter subgroup is a nilpotent self-normalizing subgroup ([1, VI.12], [2, III.4.5]). In soluble groups, Carter subgroups are $\mathfrak{N}$-projectors, they exist and are conjugate. An insoluble group may have no Carter subgroups, but by E. P. Vdovin theorem [12] Carter subgroups are conjugate whenever they exist.

Lemma 3 ([13, Theorem 15.1]). *Let $\mathfrak{F}$ be a formation. A subgroup H of a soluble group G is an $\mathfrak{F}$-projector of G if and only if $H \in \mathfrak{F}$ and H is $\mathfrak{F}$-abnormal in G.*

If $G \notin \mathfrak{F}$, but every proper subgroup of G belongs to $\mathfrak{F}$, then G is a minimal non-$\mathfrak{F}$-group. A minimal non-$\mathfrak{N}$-group is also called a Schmidt group, and its properties is well known [14].

Lemma 4 ([15, Lemma 3]). *Let $\mathfrak{F}$ be a subgroup-closed saturated formation. A soluble minimal non-$\mathfrak{F}$-group G is a group of one of the following types:*
(1) *G is a group of order p for a prime $p \notin \pi(\mathfrak{F})$;*
(2) *G is a Schmidt group.*

Lemma 5. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing all nilpotent groups. A soluble group G belongs $\mathfrak{F}$ if and only if every primary cyclic subgroup of G is $\mathfrak{F}$-subnormal.*

Proof. Assume that $G \in \mathfrak{F}$. Then every proper, and thus every primary cyclic subgroup of G, is $\mathfrak{F}$-subnormal.

Conversely, suppose that there are groups not in $\mathfrak{F}$, in which every primary cyclic subgroup is $\mathfrak{F}$-subnormal. Choose a group G of minimal order among these groups. Then every proper subgroup of G belongs to $\mathfrak{F}$. In view of Lemma 4, G is a Schmidt group, and $G = P \rtimes \langle y \rangle$ [14, Theorem 1.1]. By [14, Theorem 1.5 (5.2)], either $G^{\mathfrak{F}} \leq \Phi(G)$ or $P \leq G^{\mathfrak{F}}$. If $G^{\mathfrak{F}} \leq \Phi(G)$, then $G \in \mathfrak{F}$ since $\mathfrak{F}$ is a saturated formation, a contradiction. Let $P \leq G^{\mathfrak{F}}$. By the choice of G, $\langle y \rangle$ is $\mathfrak{F}$-subnormal in G, and so in G, there is a maximal subgroup M containing $\langle y \rangle$ and $G^{\mathfrak{F}}$, a contradiction. □

3 The Theorem Proof

Proof. Assume that every primary cyclic subgroup of a soluble group $G \notin \mathfrak{F}$ is self-normalizing or $\mathfrak{F}$-subnormal. Then according to Lemma 5, there is a cyclic p-subgroup $\langle x\rangle$ for some $p \in \pi(G)$, which is not $\mathfrak{F}$-subnormal in G. By the choice of G, $\langle x\rangle$ is self-normalizing, and so $\langle x\rangle$ is a Sylow subgroup and a Carter subgroup of G. Since a Carter subgroup is an $\mathfrak{N}$-projector [3, 5.27], we get $G = G^{\mathfrak{N}}\langle x\rangle$. In view of [1, IV.2.6], in G there is a normal Hall p'-subgroup $G_{p'}$ and $G = G^{\mathfrak{N}}\langle x\rangle = G_{p'} \rtimes \langle x\rangle$. Hence $G_{p'} \leq G^{\mathfrak{N}}$, but $G/G_{p'} \simeq \langle x\rangle \in \mathfrak{A} \subseteq \mathfrak{N}$ and $G^{\mathfrak{N}} \leq G' \leq G_{p'}$. Thus, $G_{p'} = G^{\mathfrak{N}} = G'$ and $G = G' \rtimes \langle x\rangle$. As Carter subgroups of soluble groups are conjugate [3, 5.28], we conclude that $G' \rtimes \langle x^p\rangle$ has no self-normalizing primary cyclic subgroup. Therefore $G' \rtimes \langle x^p\rangle \in \mathfrak{F}$ by Lemma 5. Thus (3) follows from (1).

Now we prove that (3) implies (2). Assume that a soluble group $G \notin \mathfrak{F}$ is represented in the form $G = G' \rtimes \langle x\rangle$, where $\langle x\rangle$ is a Sylow p-subgroup for some $p \in \pi(G)$ and a Carter subgroup, $G' \rtimes \langle x^p\rangle \in \mathfrak{F}$. Choose a subgroup H of G. If $|\langle x\rangle|$ divides $|H|$, then $\langle x\rangle^g \leq H$ for some $g \in G$ and H is self-normalizing. Suppose that $|\langle x\rangle|$ does not divide $|H|$. Then $A = G'H$ is a proper subgroup of G, and $A \in \mathfrak{F}$ by the choice of G. We conclude from $\mathfrak{A} \subseteq \mathfrak{N} \subseteq \mathfrak{F}$ that $G^{\mathfrak{F}} \leq G' \leq A$, and A is $\mathfrak{F}$-subnormal in G by Lemma 1 (4). Hence H is $\mathfrak{F}$-subnormal in G in view of Lemma 1 (5). Thus, (2) follows from (3).

Finally, assume that every proper subgroup of G is self-normalizing or $\mathfrak{F}$-subnormal. Obviously, every primary cyclic subgroup of G is also self-normalizing or $\mathfrak{F}$-subnormal. Thus (2) implies (1). □

Note that in view of Lemma 2 (1), if $\mathfrak{F}$ is a subgroup-closed formation containing all nilpotent subgroups, then every $\mathfrak{F}$-abnormal subgroup is self-normalizing. Hence the proved theorem extends results of [5, 16–18]. In particular,

Corollary. *If $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups, then for a soluble group $G \notin \mathfrak{F}$, the following statements are equivalent.*

(1) *Every primary cyclic subgroup of G is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal.*

(2) *Every proper subgroup of G is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal.*

(3) $G = G' \rtimes \langle x\rangle$, *where $\langle x\rangle$ is a Sylow p-subgroup for some $p \in \pi(G)$ and an $\mathfrak{F}$-projector of G, $G' = G^{\mathfrak{F}}$ and $G' \rtimes \langle x^p\rangle \in \mathfrak{F}$.*

Proof. Firstly, we prove that (3) follows from (1). Assume that every primary cyclic subgroup of G is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal. Then it follows from Lemma 2 (1) that every primary cyclic subgroup of G is $\mathfrak{F}$-subnormal or self-normalizing, and we can use the proved theorem. So, $G = G' \rtimes \langle x\rangle$, where $\langle x\rangle$ is a Sylow p-subgroup for some $p\in\pi(G)$ and a Carter subgroup, $G'\rtimes\langle x^p\rangle\in\mathfrak{F}$. To prove that $\langle x\rangle$ is $\mathfrak{F}$-abnormal in G, we suppose that is not true. Then $\langle x\rangle$ is $\mathfrak{F}$-subnormal in G by the choice of G. Hence every primary cyclic subgroup of G is $\mathfrak{F}$-subnormal and $G\in\mathfrak{F}$ by Lemma 5, a contradiction. Thus $\langle x\rangle$ is $\mathfrak{F}$-abnormal in G and an $\mathfrak{F}$-projector of G in view of Lemma 3. Therefore $G=G^{\mathfrak{F}}\langle x\rangle$ and $G'=G^{\mathfrak{F}}$.

Now assume that (3) is true. According to Lemma 3, we deduce that $\langle x\rangle$ is $\mathfrak{F}$-abnormal in G. By the proved theorem, every subgroup of G is self-normalizing or $\mathfrak{F}$-subnormal. Let H be a self-normalizing and non-$\mathfrak{F}$-subnormal subgroup of G. If $A=G'H$ is a proper subgroup of G, then $A\in\mathfrak{F}$ and H is $\mathfrak{F}$-subnormal in G by Lemma 1 (5), a contradiction. Hence $G=G'\rtimes\langle x\rangle=G'H$ and $\langle x\rangle\leq H$. In view of Lemma 2 (2), we obtain H is $\mathfrak{F}$-abnormal in G. Thus (3) implies (2).

Finally, assume that every proper subgroup of G is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal. Then every primary cyclic subgroup of G is also $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal. Thus (1) follows from (2).

□

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