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Finite groups with two supersoluble subgroups

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Abstract. Let G be a finite group. In this paper we obtain some sufficient conditions for the supersolubility of G with two supersoluble non-conjugate subgroups H and K of prime index, not necessarily distinct. It is established that the supersoluble residual of such a group coincides with the nilpotent residual of the derived subgroup. We prove that G is supersoluble in the following cases: one of the subgroups H or K is nilpotent; the derived subgroup G' of G is nilpotent; $|G:H| = q > r = |G:K|$ and H is normal in G. Also the supersolubility of G with two non-conjugate maximal subgroups M and V is obtained in the following cases: all Sylow subgroups of M and of V are seminormal in G; all maximal subgroups of M and of V are seminormal in G.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notation and terminology of [9, 15].

In 1953 Huppert [8] gave an example of a non-supersoluble group with three supersoluble non-conjugate subgroups of index 2. A minimal non-supersoluble group of order $2 \cdot 3 \cdot 7^2$ has supersoluble subgroups of indices 2 and 3, see [5, Lemma 3.4.3]. Sufficient conditions for the supersolubility of $G = AB$ with generalized normal subgroups A and B are obtained in [1, 2, 7, 11, 17–19]. Complete information on the groups factorized by mutually permutable subgroups is presented in the monograph [3]. Soluble groups with two supersoluble non-conjugate maximal subgroups are studied in [12]. Groups with three supersoluble subgroups whose indices are pairwise relatively prime are investigated in [4, 10, 13, 23].

In Section 3 of the present paper we obtain some sufficient conditions for the supersolubility of G with two supersoluble non-conjugate subgroups H and K of prime index, not necessarily distinct. It is established that the supersoluble residual of such a group coincides with the nilpotent residual of the derived subgroup. We prove that G is supersoluble in the following cases:

- one of the subgroups H or K is nilpotent,
- the derived subgroup G' of G is nilpotent,
- $|G:H| = q > r = |G:K|$ and H is normal in G.

These results are used in the Section 4 to obtain the supersolubility of G with two non-conjugate maximal subgroups M and V in the following cases:

- all Sylow subgroups of M and V are seminormal in G,
- all maximal subgroups of M and V are seminormal in G.

2 Preliminaries

We use G', $Z(G)$, $\Phi(G)$ and $F(G)$ to denote the derived subgroup, center, Frattini and Fitting subgroups of G, respectively.

Let G have order $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 > p_2 > \cdots > p_k$. We say that G has an ordered Sylow tower of supersoluble type if there exists a series

$$1 = G_0 < G_1 < G_2 < \cdots < G_{k-1} < G_k = G$$

of normal subgroups of G such that G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for each $i = 1, 2, \ldots, k$.

The notation $G = A \rtimes B$ is used for a semidirect product with a normal subgroup A.

Lemma 2.1 ([9, Theorem VI.9.1]). *The following statements hold:*

(1) *Every minimal normal subgroup of a supersoluble group has prime order.*

(2) *Let N be a normal subgroup of G and assume that G/N is supersoluble. If N is either cyclic, or $N \le Z(G)$, or $N \le \Phi(G)$, then G is supersoluble.*

(3) *Every supersoluble group has an ordered Sylow tower of supersoluble type.*

(4) *The derived subgroup of a supersoluble group is nilpotent.*

The commutator $[A, B]$ of arbitrary subgroups A and B of G is defined by

$$[A, B] = \langle [a, b] : a \in A, b \in B \rangle.$$

It is clear that $[A, B] \le G'$ and $G' = [G, G]$.

Lemma 2.2 ([15, Lemma 4.8], [19, Lemma 4]). *Let $G = AB$. Then:*

(1) $[A, B]$ *is normal in G,*

(2) $[A[A, B]/[A, B], B[A, B]/[A, B]] = 1$,

(3) *if A_1 is normal in A, then $A_1[A, B]$ is normal in G,*

(4) $G' = A'B'[A, B]$,

(5) *if A and B are normal in G and $(|G : A|, |G : B|) = 1$, then $G' = A'B'$.*

The formations of all abelian, nilpotent and supersoluble groups are denoted by $\mathfrak{A}$, $\mathfrak{N}$ and $\mathfrak{U}$, respectively. Let $\mathfrak{F}$ be a formation. The subgroup

$$G^{\mathfrak{F}} = \bigcap\{N \lhd G : G/N \in \mathfrak{F}\}$$

is called the $\mathfrak{F}$-residual of G. The subgroups $G^{\mathfrak{A}}$, $G^{\mathfrak{N}}$ and $G^{\mathfrak{U}}$ are called abelian, nilpotent and supersoluble residual of G, respectively. It is clear that the abelian residual of G coincides with the derived subgroup of G: $G^{\mathfrak{A}} = G'$. We define $\mathfrak{F}\mathfrak{H} = \{G \in \mathfrak{E} : G^{\mathfrak{H}} \in \mathfrak{F}\}$ and call $\mathfrak{F}\mathfrak{H}$ the formation product of $\mathfrak{F}$ and $\mathfrak{H}$. Here $\mathfrak{E}$ is the class of all finite groups. As usually, $\mathfrak{F}^2 = \mathfrak{F}\mathfrak{F}$ and $\mathfrak{F}^n = \mathfrak{F}^{n-1}\mathfrak{F}$ for every natural $n \geq 3$. By Lemma 2.1 (4), for example, we have $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$.

Lemma 2.3 ([19, Lemma 6]). *Let G be a soluble group. Assume that $G \notin \mathfrak{U}$, but $G/K \in \mathfrak{U}$ for every non-trivial normal subgroup K of G. Then:*

(1) *G contains a unique minimal normal subgroup N, $N = F(G) = \mathrm{O}_p(G) = C_G(N)$ for some $p \in \pi(G)$,*

(2) *$Z(G) = \mathrm{O}_{p'}(G) = \Phi(G) = 1$,*

(3) *G is primitive; $G = N \rtimes M$, where M is maximal in G with trivial core,*

(4) *N is an elementary abelian subgroup of order p^n, $n > 1$,*

(5) *if V is a subgroup G and $G = VN$, then $V = M^x$ for some $x \in G$.*

Lemma 2.4 ([15, Lemma 5.8, Theorem 5.11]). *Let $\mathfrak{F}$ and $\mathfrak{H}$ be formations, and let K be normal in G. Then:*

(1) *$(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$,*

(2) *$G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$,*

(3) *if $\mathfrak{H} \subseteq \mathfrak{F}$, then $G^{\mathfrak{F}} \leq G^{\mathfrak{H}}$.*

The smallest non-negative integer n such that $G^{(n)} = 1$ is called the derived length of G and denoted by $d(G)$. Here $G^{(m)} = (G^{(m-1)})'$ is the m-th derived subgroup of G. By the definition of the formation product, we obtain $d(G) = m$ if and only if $G \in \mathfrak{A}^m \setminus \mathfrak{A}^{m-1}$.

Lemma 2.5. *If $d(G/\Phi(G)) \leq 2$, then $G \in \mathfrak{N}\mathfrak{A}$.*

Proof. If $d(G/\Phi(G)) \leq 2$, then there is an abelian normal subgroup $A/\Phi(G)$ of $G/\Phi(G)$ and G/A is abelian. By [15, Theorem 3.24], A is nilpotent, hence we have $G \in \mathfrak{N}\mathfrak{A}$. □

Recall that a group G is said to be siding if every subgroup of the derived subgroup G' is normal in G, see [20, Definition 2.1]. Metacyclic groups, T-groups (groups in which every subnormal subgroup is normal) are siding. The group $G = (Z_6 \times Z_2) \rtimes Z_2$ (IdGroup(G) = [24,8], [26]) is siding, but it is not metacyclic and is not a T-group.

Lemma 2.6. *Let G be siding. Then the following statements hold:*

(1) *if N is normal in G, then G/N is siding,*

(2) *if H is a subgroup of G, then H is siding,*

(3) *G is supersoluble.*

Proof. (1) By [15, Lemma 4.6], $(G/N)' = G'N/N$. Let $\overline{A} = A/N$ be an arbitrary subgroup of $(G/N)'$. Then

$$A \leq G'N, \quad A = A \cap G'N = (A \cap G')N.$$

Since $A \cap G' \leq G'$, we have $A \cap G'$ is normal in G. Hence $\overline{A} = (A \cap G')N/N$ is normal in G/N.

(2) Since $H \leq G$, it follows that $H' \leq G'$. Let A be an arbitrary subgroup of H'. Then $A \leq G'$ and A is normal in G. Therefore, A is normal in H.

(3) We proceed by induction on the order of G. Let $N \leq G'$ and $|N| = p$, where p is prime. By the hypothesis, N is normal in G. By induction, G/N is supersoluble and G is supersoluble by Lemma 2.1 (2). □

Remark 2.7. By Lemma 2.6, the class of all siding groups is a hereditary homomorph. The supersoluble group $G = S_3 \times S_3$ (IdGroup(G) = [36,10])) is not siding. Really, the derived subgroup $G' = \langle a \rangle \times \langle b \rangle$ is an elementary abelian group of order 9, but the subgroup $\langle ab \rangle$ of G' is not normal in G. Moreover, all primitive quotients of G are isomorphic to either a cyclic group of order 2, or S_3, hence are siding. Hence the class of all siding groups is not a Schunck class and formation.

3 Supersolubility of a group G with a pair of non-conjugate supersoluble subgroups H and K of prime index

Lemma 3.1. *If G has a supersoluble subgroup H of prime index, then G/H_G is supersoluble.*

Proof. Suppose that $|G : H| = r$ is prime. Using induction on the order of G, we first show that G is soluble. If N is a non-trivial normal subgroup in G, then either $N \leq H$, or $G = HN$. If $N \leq H$, then G/N is soluble by induction, hence G is soluble. If $G = HN$, then $G/N \simeq H/H \cap N$ is supersoluble and $H \cap N$ is a supersoluble subgroup of prime index r in N. By induction, N is soluble, therefore, G is soluble. Consequently, G is simple and $r > 2$.

Let R be a Sylow r-subgroup of G and let $H_{r'}$ be an r'-Hall subgroup of H. Since $G = H_{r'}R$, by [24, Theorem 1], it follows that G is isomorphic to $\mathrm{PSL}(2, q)$, $q \equiv -1 \pmod 4$, q is a prime. However, it does not have a supersoluble subgroup of prime index, see [9, Theorem II.8.27]. Thus, G is soluble and G/H_G is primitive. Hence

$$G/H_G = N/H_G \rtimes H/H_G,$$

$$|N/H_G| = |G/H_G : H/H_G| = r,$$

$$N/H_G = C_{G/H_G}(N/H_G),$$

H/H_G is cyclic and G/H_G is supersoluble. □

Remark 3.2. A group with two non-conjugate subgroups H and K of prime index is soluble if one of the subgroups H or K is supersoluble, and the other is soluble. The group $\mathrm{PSL}(2, 7)$ has non-conjugate subgroups of orders 24 (each of them isomorphic to the symmetric group S_4 of degree 4) and their indices are equal to 7.

Lemma 3.3. *Suppose that G has non-conjugate supersoluble subgroups H and K of prime index. If G is non-supersoluble, then the following statements hold:*

(1) $F(G) \leq H \cap K$,

(2) *G has an ordered Sylow tower of supersoluble type,*

(3) *H or K is normal in G.*

Proof. Let $|G : H| = q \geq r = |G : K|$, where q and r are prime numbers.

(1) By Lemma 3.1, G is soluble. Since $\Phi(G) \leq H \cap K$ and $F(G/\Phi(G)) = F(G)/\Phi(G)$, it follows that $\Phi(G) = 1$. Suppose that $F(G) \not\leq H \cap K$. Since $F(G)$ is the product of minimal normal subgroups of G, there is a minimal normal subgroup N of G such that $N \not\leq H \cap K$. Let $N \not\leq K$. Then $G = N \rtimes K$ and $|N| = |G : K|$ is prime. Since K is supersoluble, G is supersoluble by Lemma 2.1, a contradiction. Similarly, if $N \not\leq H$, then $G = N \rtimes H$, $|N| = |G : H| = q$ and G is supersoluble, a contradiction. Hence $F(G) \leq H \cap K$.

(2) Since $G = HK$, it follows that $G_p = H_p K_p$ for some Sylow p-subgroups G_p, H_p and K_p of G, H and K, respectively. Let p be the greatest prime in $\pi(G)$. By the hypothesis, H and K are supersoluble, hence H_p and K_p are normal in H and K, respectively. Since $p \geq q$ and $p \geq r = |G : K|$, we have that H_p and K_p are normal in G by Sylow's theorem, so G is p-closed and

$$G_p = H_p K_p \leq F(G) \leq H \cap K.$$

If G/G_p is non-supersoluble, then by induction, it has an ordered Sylow tower of supersoluble type. If G/G_p is supersoluble, then it also has an ordered Sylow tower of supersoluble type. Hence G has an ordered Sylow tower of supersoluble type.

(3) We use induction on the order of G. Let N be a minimal normal subgroup of G, $N \leq H \cap K$. If G/N is non-supersoluble, then by induction, H/N or K/N is normal in G/N, hence H or K is normal in G. Therefore, we assume that G/N is supersoluble for every non-trivial normal subgroup N of G, which is contained in $H \cap K$. Since $\Phi(G) < F(G) \leq H \cap K$, we have that $\Phi(G) = 1$ and $F(G)$ is a minimal normal subgroup of G. By (2), $F(G) = G_p$ is a Sylow p-subgroup of G and p is the greatest prime in $\pi(G)$. Since $F(G) = C_G(F(G)) \leq H$ and H is supersoluble, it follows that $H' \leq F(H) = F(G)$ and $H/F(G)$ is abelian. Similarly, $K/F(G)$ is abelian. If $H/F(G)$ and $K/F(G)$ are non-normal in $G/F(G)$, then $H/F(G)$ and $K/F(G)$ are self-normalizing and conjugate in $G/F(G)$ as Carter subgroups. In this case the subgroups H and K are conjugate, a contradiction. Therefore, one of the subgroups $H/F(G)$ or $K/F(G)$ is normal in $G/F(G)$, and therefore, one of the subgroups H or K is normal in G. □

Theorem A. *Suppose that G has non-conjugate subgroups H and K of prime index. If H is nilpotent and K is supersoluble, then G is supersoluble.*

Proof. We proceed by induction on $|G|$. Suppose that G is not supersoluble of minimal order. By Lemma 3.1, G is soluble and $G_p \leq F(G) \leq H \cap K$ by Lemma 3.3 for the greatest $p \in \pi(G)$. Since $\Phi(G) \leq H \cap K$ and $F(G/\Phi(G)) = F(G)/\Phi(G)$, it follows, by Lemma 2.1 (2), that $\Phi(G) = 1$ and

$$F(G) = N_1 \times N_2 \times \cdots \times N_n,$$

where N_i is a minimal normal subgroup of G for every i. By induction, G/N_i is supersoluble. If $n > 1$, then $G \simeq G/(N_1 \cap N_2)$ is supersoluble, a contradiction. Hence we have $n = 1$ and $F(G) = G_p$ is a minimal normal subgroup of G. Since $F(G) = C_G(F(G))$, we have $H = F(G)$ and $H \leq K$ by Lemma 3.3 (1), a contradiction. □

Corollary 3.4. *If G has non-conjugate nilpotent subgroups H and K of prime index, then G is supersoluble.*

Theorem B. *Suppose that G has non-conjugate subgroups H and K of prime index. If H is supersoluble, K is normal in G and is siding, then G is supersoluble.*

Proof. Suppose that G is non-supersoluble and we proceed by induction on the order of G. Since K' is normal in G and is nilpotent, $K' \leq H$ by Lemma 3.3 (1). If $K' = 1$, then K is abelian and G is supersoluble by Theorem A, a contradiction.

Therefore, $K' \neq 1$ and there is a subgroup N, which is normal in H, and of prime order, which is contained in K'. Since K is siding, N is normal in K, hence N is normal in G. By Lemma 2.6 (1), the hypotheses of the theorem are inherited by all quotients of G. By induction, G/N is supersoluble, so G is supersoluble. □

Corollary 3.5. *Suppose that G has non-conjugate subgroups H and K of prime index. If H and K are siding, then G is supersoluble.*

Proof. Suppose that G is non-supersoluble. Then by Lemma 3.3 (3), one of the subgroups H or K is normal in G. As a consequence of Theorem B, G is supersoluble, a contradiction. □

Theorem C. *If G has non-conjugate supersoluble subgroups H and K of prime index, then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = [H, K]^{\mathfrak{N}}$.*

Proof. If G is supersoluble, then $G^{\mathfrak{U}} = 1$ and G' is nilpotent by Lemma 2.1 (4). Therefore, $(G')^{\mathfrak{N}} = [H, K]^{\mathfrak{N}} = 1$ and the statement is true. Further, we assume that G is non-supersoluble. By Lemma 3.1, G is soluble. First we prove that $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$.

Since $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$, it follows that

$$G^{(\mathfrak{N}\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}} \leq G^{\mathfrak{U}}$$

by Lemma 2.4 (2)–(3). Next we check the reverse inclusion. For this we prove that $G/(G')^{\mathfrak{N}}$ is supersoluble. The derived subgroup

$$(G/(G')^{\mathfrak{N}})' = G'(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} = G'/(G')^{\mathfrak{N}}$$

is nilpotent. Since H is a maximal subgroup of G, we have either $(G')^{\mathfrak{N}} \leq H$, or $H(G')^{\mathfrak{N}} = G$. Let $H(G')^{\mathfrak{N}} = G$. Then we have that $G/(G')^{\mathfrak{N}}$ is supersoluble and $G^{\mathfrak{U}} \leq (G')^{\mathfrak{N}}$. Hence $(G')^{\mathfrak{N}} \leq H \cap K$. The derived subgroup $(G/(G')^{\mathfrak{N}})'$ is nilpotent in $G/(G')^{\mathfrak{N}}$,

$$(G/(G')^{\mathfrak{N}})' \leq F(G/(G')^{\mathfrak{N}}),$$

the subgroups $H/(G')^{\mathfrak{N}}$ and $K/(G')^{\mathfrak{N}}$ are supersoluble, non-conjugate and have prime indices. If $(G/(G')^{\mathfrak{N}})'$ is not contained in $(H/(G')^{\mathfrak{N}}) \cap (K/(G')^{\mathfrak{N}})$, then $F(G/(G')^{\mathfrak{N}})$ is not contained in $(H/(G')^{\mathfrak{N}}) \cap (K/(G')^{\mathfrak{N}})$ and by Lemma 3.3 (1), $G/(G')^{\mathfrak{N}}$ is supersoluble. If

$$(G/(G')^{\mathfrak{N}})' \leq (H/(G')^{\mathfrak{N}}) \cap (K/(G')^{\mathfrak{N}},$$

then $H/(G')^{\mathfrak{N}}$ and $K/(G')^{\mathfrak{N}}$ are normal in $G/(G')^{\mathfrak{N}}$ and $G/(G')^{\mathfrak{N}}$ is supersoluble by Baer theorem [2]. In any case $G/(G')^{\mathfrak{N}}$ is supersoluble and $G^{\mathfrak{U}} \leq (G')^{\mathfrak{N}}$. Thus, the equality $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$ is proved.

By Lemma 3.3 (3), one of the subgroups H or K is normal in G. We assume that H is normal in G. Then $[H,K] \leq H$ and H' is a normal nilpotent subgroup in G. Hence $H' \leq F(G) \leq H \cap K$ and $H'K'$ is a normal nilpotent subgroup in K. Let $V = [H,K]^{\mathfrak{N}}$. Since $V \leq [H,K] \leq H$, V is nilpotent and $V \leq K$, hence $H'K'V$ is a normal nilpotent subgroup in K. The quotient

$$\begin{aligned} G/V &= (H/V)(K/V), \\ (G/V)' &= (H/V)'(K/V)'([H/V, K/V]) \\ &= (H'V/V)(K'V/V)([H,K]/V) \\ &= ((H'K'V)/V)([H,K]/V) \end{aligned}$$

by Lemma 2.2 (4). The subgroup $[H,K]/V$ is normal and nilpotent in G/V. If $[H,K]/V \leq K/V$, then $(G/V)'$ is nilpotent. Since the equality $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$ is proved, it follows that G/V is supersoluble. If $[H,K]/V$ is not contained in K/V, then G/V is supersoluble by Lemma 3.3 (1). So G/V is supersoluble in any case and we obtain $G^{\mathfrak{U}} \leq V = [H,K]^{\mathfrak{N}}$. Since $[H,K]^{\mathfrak{N}} \leq (G')^{\mathfrak{N}} = G^{\mathfrak{U}}$, we have $G^{\mathfrak{U}} = [H,K]^{\mathfrak{N}}$. □

Corollary 3.6. *Suppose that G has non-conjugate supersoluble subgroups H and K of prime index. Then the following statements are equivalent:*

(1) *G is supersoluble.*

(2) *G' is nilpotent.*

(3) *$[H,K]$ is nilpotent.*

Proof. Suppose that G is supersoluble. By Lemma 2.1 (4), the derived subgroup G' is nilpotent. Since $[H,K] \leq G'$, statements (2) and (3) follow from (1). Let $[H,K]$ be nilpotent. Then G is supersoluble by Theorem C and G' is nilpotent by Lemma 2.1 (4). Therefore, (1) and (2) follow from (3). If G' is nilpotent, then G supersoluble by Theorem C. Consequently, (1) follows from (2) and all three statements are equivalent. □

Corollary 3.7. *Suppose that G has non-conjugate supersoluble subgroups H and K of prime index. Then G is supersoluble if and only if $d(G/\Phi(G)) \leq 2$.*

Proof. If G is supersoluble, then its derived subgroup is nilpotent, hence we have $G' \leq F(G)$. Since $F(G)/\Phi(G)$ is abelian, we conclude that $d(G/\Phi(G)) \leq 2$.

Conversely, suppose that G has non-conjugate supersoluble subgroups H and K of prime index and $d(G/\Phi(G)) \leq 2$. By Lemma 2.5, $G \in \mathfrak{N}\mathfrak{A}$, hence its derived subgroup is nilpotent. By Corollary 3.6, G is supersoluble. □

Corollary 3.8. *Suppose that G has non-conjugate supersoluble subgroups H and K of prime index. If G is non-supersoluble, then $d(G/\Phi(G)) = 3$.*

Proof. By Lemma 3.3 (3), one of the subgroups H or K is nilpotent in G. Let H be normal in G. Then H' is normal and nilpotent in G. Hence $d(G/F(G)) \leq 2$. Since $F(G)/\Phi(G)$ is abelian, we have $d(G/\Phi(G)) \leq 3$. By Corollary 3.7, we have $d(G/\Phi(G)) = 3$. □

Theorem D. *Let H and K be supersoluble subgroups of G,*

$$|G : H| = q > r = |G : K|,$$

where q and r are the prime numbers. If H is normal in G, then G is supersoluble.

Proof. Suppose that G is non-supersoluble. Since H is normal in G and G/H is cyclic, G is soluble. By Lemma 3.3 (2), G has an ordered Sylow tower of supersoluble type, hence G has a normal subgroup U of index t for the smallest $t \in \pi(G)$. If $r = t$, then K is normal in G and G is supersoluble by [11]. Let $r > t$. Then

$$G = HU = KU, \quad q, r \in \pi(U),$$
$$q = |G : H| = |U : U \cap H| > r = |G : K| = |U : U \cap K|,$$

the subgroups $U \cap H$ and $U \cap K$ are supersoluble and $U \cap H$ is normal in U. By induction, U is supersoluble. Since $G = HU$, G is supersoluble by [11]. □

Corollary 3.9. *Let H and K be supersoluble subgroups of G,*

$$|G : H| = q > r = |G : K|,$$

where q and r are prime numbers. If G is metanilpotent, then G is supersoluble.

Proof. Suppose that G is non-supersoluble. By Lemma 3.3 (1), $F(G) \leq H \cap K$. Since G is metanilpotent, we have $G^{\mathfrak{N}} \leq F(G)$. Hence $G/F(G)$ is nilpotent. Therefore, H and K are normal in G. Hence G is supersoluble by Theorem D. □

4 Groups with some seminormal subgroups in maximal subgroups

A subgroup X of G is called *seminormal* in G, if there exists a subgroup Y such that $G = XY$ and XY_1 is a proper subgroup in G for every proper subgroup Y_1 of Y. Here such a subgroup Y is called a *supersupplement* to X in G.

Groups with some seminormal subgroups have been investigated in the works of many authors, see, for example, [6, 14, 16, 21, 22, 25]. We need the following properties.

Lemma 4.1 ([14, Lemma 2]). *The following statements hold:*

(1) *If H is a seminormal subgroup of G and $H \leq X \leq G$, then H is seminormal in X.*

(2) *If H is a seminormal subgroup of G and $N \lhd G$, then HN/N is seminormal in G/N.*

(3) *If H is a seminormal subgroup of G and K is a supersupplement to H in G, then H is permutable with subgroup L^g for all $L \leq K$ and all $g \in G$. In particular, K^g is a supersupplement to H in G for every $g \in G$.*

Lemma 4.2. *(*[14, Lemma 10]*) If A is a seminormal subgroup of G, then A^G is soluble in each of the following cases:*

(1) *A is 2-nilpotent,*

(2) *A is soluble and 3 does not divide the order of A.*

Lemma 4.3. *The following statements hold:*

(1) ([21, Theorem 2]) *Let p be the greatest prime divisor of the order of G and let P be a Sylow p-subgroup of G. If P is seminormal in G, then P is normal in G.*

(2) ([25, Lemma 1]) *If any Sylow subgroup of G is seminormal in G, then G is supersolvable.*

Lemma 4.4 ([21, Corollary 1]). *A group G is supersoluble if and only if every maximal subgroup of G is seminormal in G.*

Lemma 4.5. *Let M be a maximal subgroup of G. If all Sylow subgroups of M are seminormal in G, then M and G/M_G is supersoluble.*

Proof. Let M_r be an arbitrary Sylow r-subgroup of M, $r \in \pi(M)$. By Lemma 4.1, M_r is seminormal in M and by Lemma 4.3 (2), M is supersoluble. By Lemma 4.2, M_r^G is soluble and

$$M \leq H = \prod_{r \in \pi(M)} M_r^G.$$

The subgroup H is normal in G and is soluble. If $M = H$, then $|G : M|$ is prime and G is soluble. If $H = G$, then G is soluble.

We use induction on the order of G and prove that G/M_G is supersoluble. If $M_G \neq 1$, then the quotient M/M_G is a maximal subgroup of G/M_G. If $\overline{P}$ is a Sylow p-subgroup of M/M_G, then $\overline{P} = PM_G/M_G$ for some Sylow p-subgroup P of M. By hypothesis, P is seminormal in G and by Lemma 4.1, a subgroup $PM_G/M_G = \overline{P}$ is seminormal in G/M_G. Consequently, for G/M_G and its maximal subgroup M/M_G all conditions of the lemma are satisfied. By induction, $(G/M_G)/(M/M_G)_{G/M_G}$ is supersoluble and since $(M/M_G)_{G/M_G} = 1$, we have that G/M_G is supersoluble. Therefore, we consider that $M_G = 1$, i.e. G is primitive:

$$G = N \rtimes M, \quad N = \mathrm{O}_p(G) = F(G) = C_G(N),$$

N is the unique minimal normal subgroup in G. If N is a Sylow subgroup of G, then all Sylow subgroups of G are seminormal in G and G is supersoluble by Lemma 4.3 (2). Hence p divides the order of M. Since $\mathrm{O}_p(G/N) = 1$, we have that $\mathrm{O}_p(M) = 1$ and each Sylow p-subgroup M_p of M is non-normal in M. Since M is supersoluble, a Sylow q-subgroup M_q is normal in M for the greatest $q \in \pi(M)$. Hence $q > p$ and M_q is a Sylow q-subgroup of G and q is the greatest prime in $\pi(G)$. Since M_q is seminormal in G, M_q is normal in G by Lemma 4.3 (1) and $M_q \leq M_G = 1$, a contradiction. □

Theorem E. *Suppose that G has non-conjugate maximal subgroups H and K. If all Sylow subgroups of H and of K are seminormal in G, then G is supersoluble.*

Proof. By Lemma 4.5, G is soluble and the quotients G/H_G and G/K_G are supersoluble, in particular, the indices of subgroups H and K in G are the prime numbers. By Lemma 4.3 (2), H and K are supersoluble.

Suppose that G is non-supersoluble; we use induction on the order of G. By Lemma 3.3, we have that G has an ordered Sylow tower of supersolvable type and $F(G) \leq H \cap K$. Let P be a Sylow p-subgroup of G for the greatest $p \in \pi(G)$ and $N \leq Z(P)$, $|N| = p$. Let R be a Sylow r-subgroup of G, $r \neq p$. Since $G = HK$, it follows that $R = H_r K_r$ for some Sylow r-subgroups H_r and K_r of H and K, respectively. By hypothesis, H_r is seminormal in G, hence there exists a subgroup U of G such that $G = H_r U$ and H_r is permutable with any subgroup of U. Since P is normal in G, $N \leq P \leq U$ and H_r is permutable with N. Similarly, K_r is permutable with N. Therefore, R is permutable with N. This is true for any $r \neq p$, thus $G_{p'}$ is permutable with N. Currently, $G_{p'}N$ is a subgroup of G and N is normal in $G_{p'}N$. Since $N \leq Z(P)$, it follows that N is normal in $PG_{p'} = G$. As $N \leq F(G) \leq H \cap K$, we have $G/N = (H/N)(K/N)$, H/N and K/N are non-conjugate maximal subgroups of G/N. Let $\overline{Q}$ be a Sylow q-subgroup of H/N. Then H has a Sylow q-subgroup Q such that $\overline{Q} = QN/N$. By hypothesis, Q is seminormal in G. By Lemma 4.1, $\overline{Q} = QN/N$ is seminormal in G/N. Similarly, every Sylow subgroup of K/N is seminormal in G/N. By induction, G/N is supersoluble, and by Lemma 2.1 (2), G is supersoluble. □

Example 4.6. The subgroup $G = \mathrm{PSL}(2, 5)$ has the maximal subgroups

$$H = Z_3 \rtimes Z_2 \quad \text{and} \quad K = Z_5 \rtimes Z_2.$$

The maximal subgroups of Sylow subgroups of H and K are trivial, hence are seminormal in G, but G is non-soluble.

Lemma 4.7. *Let M be a maximal subgroup of G and suppose that all maximal subgroups of M are seminormal in G. Then G is soluble.*

Proof. Let K be a maximal subgroup of M. If $K = 1$, then $|M|$ is prime and G is soluble [9, IV.7.4]. Next, we consider the case where $K \neq 1$. By hypothesis, K is seminormal in G and, by Lemma 4.1, K is seminormal in M. Since K is an arbitrary maximal subgroup of M, by Lemma 4.4, M is supersoluble and consequently is 2-nilpotent. Then K is also 2-nilpotent and by Lemma 4.2, K^G is soluble. Since M is a maximal subgroup of G, it follows that either $MK^G = G$, or $K^G \leq M$. If $MK^G = G$, then G is soluble. Let $MK^G \neq G$. Then $K \leq K^G \leq M$ and either $K^G = M$, or $K = K^G < M$. If $K^G = M$, then $|G/M|$ is prime and G is soluble. If $K^G = K$, then $|M/K|$ is prime and G/K is soluble by [9, IV.7.4], hence G is soluble. □

Theorem F. *Let H and K be non-conjugate maximal subgroups of G. If all maximal subgroups of H and of K are seminormal in G, then G is supersoluble.*

Proof. We use induction on the order of G. By Lemma 4.7, G is soluble. By Lemma 4.1, every maximal subgroup of H is seminormal in H and by Lemma 4.4, H is supersoluble. Similarly, K is supersoluble.

Let N be an arbitrary non-trivial normal subgroup in G. Then either $HN = G$, or $HN = H$. If $HN = G$, then

$$G/N = HN/N \cong H/H \cap N$$

is supersoluble. If $HN = H$, then $N \leq H$. Similarly, either $KN = G$ and G/N is supersoluble, or $N \leq K$. Let $N \leq H \cap K$. Then G/N has non-conjugate maximal subgroups H/N and K/N. If $\overline{S}$ is a maximal subgroup of H/N, then H has a maximal subgroup S such that $\overline{S} = S/N$. By hypothesis, S is seminormal in G and by Lemma 4.1, $\overline{S} = S/N$ is seminormal in G/N. Similarly, if $\overline{T}$ is a maximal subgroup of K/N, then it is seminormal in G/N. Therefore, for G/N with non-conjugate maximal subgroups H/N and K/N the conditions of the theorem are satisfied. By induction, G/N is supersoluble.

So, in any case G/N is supersoluble. By Lemma 2.3, G is primitive:

$$G = N \rtimes M, \quad N = \mathrm{O}_p(G) = F(G) = C_G(N), \quad |N| = p^n > p, \quad M_G = 1,$$

N is the unique minimal normal subgroup of G, M is a maximal subgroup of G. If $N \not\leq H$ and $N \not\leq K$, then $G = N \rtimes H = N \rtimes K$, H and K are conjugate in M by Lemma 2.3 (5), a contradiction. Hence $N \leq H$ or $N \leq K$. Without loss of generality, we further assume that $N \leq K$.

The following cases are possible.

Case 1: $N \leq H \cap K$. In this case

$$H = N \rtimes (H \cap M),$$
$$K = N \rtimes (K \cap M),$$
$$M = (H \cap M)(K \cap M).$$

Since G/N is supersoluble, the indices $|G : H|$ and $|G : K|$ are prime numbers. By Lemma 3.3, N is a Sylow p-subgroup of G for the greatest $p \in \pi(G)$ and

$$G_p = H_p = K_p = N, \quad M = G_{p'}, \quad H \cap M = H_{p'}, \quad K \cap M = K_{p'}.$$

Let $|G : H| = q$. We choose a maximal subgroup H_1 in H such that

$$H_{p'} \leq H_1 < H.$$

Since H is supersoluble, we have $|H : H_1| = p$ and $|G : H_1| = pq$. By hypothesis, H_1 is seminormal in G. Let T is a supersupplement to H_1 in G. Then G has a subgroup $H_1 T_q$ and $|G : H_1 T_q| = p$. Hence $M = G_{p'} \leq (H_1 T_q)^g$ for some $g \in G$ and $|N| = |G : M| = p$, a contradiction. This concludes Case 1.

Case 2: $N \not\leq H$ and $N = G_p$. In this case

$$G = N \rtimes H,$$
$$H = G_{p'},$$
$$K = N \rtimes (H \cap K).$$

Let $N_1 \leq N$, $|N_1| = p$. By hypothesis, every maximal subgroup H_i of H is seminormal in G. Therefore, there are subgroups U_i such that $H_i U_i = G$ and $H_i N_1 = N_1 H_i$, since N is contained in every U_i. If H has two non-conjugate maximal subgroups H_1 and H_2, then $H_1 H_2 = H$ and H is permutable with N_1. By the maximality of the subgroup H, we have that $HN_1 = G$, a contradiction. Consequently, in H there exists a unique maximal subgroup and H is cyclic of order q^a. Hence H is a Sylow q-subgroup of G. Since G/N is a q-group, it follows that a maximal subgroup K/N has index q. Therefore, $K = N \rtimes (K \cap H)$ and $K \cap H$ is a cyclic Sylow q-subgroup of K of order q^{a-1}.

Let K_1 be a maximal subgroup of K such that $K \cap H \leq K_1 < K$. Since K is supersoluble, we have that $|K : K_1| = p$ and $|G : K_1| = pq$. By hypothesis, K_1 is seminormal in G. Let V be a supersupplement to K_1 in G. Since $G = K_1 V$, a cyclic Sylow q-subgroup G_q has the representation

$$G_q = (K \cap H)V_q, \quad |G_q| = q^a, \quad |K \cap H| = q^{a-1}.$$

Since the cyclic primary group cannot be expressed as a product of two proper subgroups, it follows that $G_q = V_q = H^x$ for some $x \in G$. Since $K_1 V_q$ is a subgroup, $K_1 V_q = K_1 H^x = H^x$ and $|G : H^x| = p$, a contradiction. This concludes Case 2.

Case 3: $N \nleq H$ and $N < G_p$. In this case

$$G = N \rtimes H,$$

$$G_p = N \rtimes (G_p \cap H), \quad G_p \cap H \neq 1,$$

$$K = N \rtimes (H \cap K).$$

Since $O_p(G/N) = 1$, we conclude that $O_p(H) = 1$. In view of the supersolubility of H, there is a prime number $r \in \pi(H)$ such that $r > p$ and H_r is normal in H. Since $N = C_G(N) \leq K$, it follows that $p \in \pi(K)$ and p is the greatest prime in $\pi(K)$. Hence $H_r = G_r$, $H = N_G(H_r)$ and r does not divide $|K|$. Since G/N is supersoluble and K/N is a maximal subgroup of G/N, we have $|G : K| = r$ and $|G_r| = r$. We choose in K the maximal subgroup K_1 such that $(H \cap K) \leq K_1 < K$. It is obvious that

$$|K : K_1| = p, \quad NK_1 = K, \quad |G : K_1| = pr.$$

By hypothesis, K_1 is seminormal in G. Let T be a supersupplement to K_1 in G. Then r divides $|T|$, $K_1 T_r$ is a subgroup and $|G : K_1 T_r| = p$. Currently,

$$NK_1 T_r = KT_r = G, \quad N \neq N \cap K_1 T_r$$

and $N \cap K_1 T_r$ is normal in G, a contradiction. This concludes Case 3. □

Example 4.8. The group A_5 has the maximal subgroups

$$H = Z_3 \rtimes Z_2 \quad \text{and} \quad K = Z_5 \rtimes Z_2.$$

All 2-maximal subgroups of H and of K are trivial, hence are seminormal in A_5, but A_5 is non-soluble.

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