Publ. Math. Debrecen
96/1-2 (2020), 141–147
DOI: 10.5486/PMD.2020.8633

On the residual of a finite group with semi-subnormal subgroups

By ALEXANDER TROFIMUK (Gomel)

Abstract. A subgroup A of a group G is called *seminormal* in G, if there exists a subgroup B such that $G = AB$ and AX is a subgroup of G for every subgroup X of B. We introduce the new concept that unites subnormality and seminormality. A subgroup A of a group G is called *semi-subnormal* in G, if A is subnormal in G or seminormal in G. In this paper, the $\mathfrak{F}$-residual of a group $G = AB$ with semi-subnormal subgroups A and B such that $A, B \in \mathfrak{F}$, where $\mathfrak{F}$ is a saturated formation and $\mathfrak{U} \subseteq \mathfrak{F}$, is studied. Here $\mathfrak{U}$ is the class of all supersoluble groups and the $\mathfrak{F}$-residual of G is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [7], [9]. The monographs [6], [9] contain the necessary information of the theory of formations.

It is well known that subgroups A and B of G permute if $AB = BA$. If H and K are subgroups of G such that H is permutable with every subgroup of K and K is permutable with every subgroup of H, we say that H and K are mutually permutable. We say that H and K are totally permutable if every subgroup of H is permutable with every subgroup of K. If $G = AB$ and the subgroups A and B are mutually (respectively totally) permutable, then G is called a mutually (respectively totally) permutable product of A and B, see [3, p. 149].

Mathematics Subject Classification: 20D10, 20D20.
Key words and phrases: finite group, residual, semi-subnormal subgroups, derived subgroup, nilpotent and supersoluble groups.

Let $\mathfrak{F}$ be a non-empty formation of groups, that is, $\mathfrak{F}$ is closed under taking homomorphic images and subdirect products. Then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual of G, that is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$. Obviously, G is supersoluble if and only if $G^{\mathfrak{U}} = 1$. Here $\mathfrak{U}$ is the formation of all supersoluble groups. A well-known theorem of Doerk and Hawkes [6, IV.1.18] states that for a formation $\mathfrak{F}$ of soluble groups, the $\mathfrak{F}$-residual respects the operation of forming direct products. The above results confirm that the $\mathfrak{F}$-residuals play an important role in the study of the structure of groups. Fortunately, these residuals have a nice behaviour in mutually (totally) permutable products.

A. Ballester-Bolinches, M. C. Pedraza-Aguilera and M. D. Perez-Ramos in [5] extended a previous result of Doerk and Hawkes by considering a totally permutable product of subgroups.

Theorem 1.1 ([5, Theorem A])**.** *Let $\mathfrak{F}$ be a formation of soluble groups such that $\mathfrak{U} \subseteq \mathfrak{F}$. If $G = AB$ is the product of the totally permutable subgroups A and B, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$.*

Asaad and Shaalan's result [2, Theorem 3.1] is a particular case of Theorem 1.1 when A and B are supersoluble.

In [3] and [4], a similar decomposition of the $\mathfrak{F}$-residual was obtained for a group that is a mutually permutable product of subgroups.

Theorem 1.2. *Let $G = AB$ be the mutually permutable product of the subgroups A and B. Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of supersoluble groups. Then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$ in each of the following cases:*

(1) *the derived subgroup G' is nilpotent, see* [4, Theorem A]*;*

(2) $(A \cap B)_G = 1$, *see* [3, Theorem 4.5.8].

The results of Asaad and Shaalan [2, Theorem 3.8] and M. Alejandre, A. Ballester-Bolinches and J. Cossey [1, Theorem 1] follow from Theorem 1.2 when A and B are supersoluble.

Without restrictions on the derived subgroup G' and the core $(A \cap B)_G$, V. S. Monakhov, I. K. Chirik and the author in [10], [11] and [12] described the structure of the $\mathfrak{U}$-residual of G, when G is a product of two either supersoluble subnormal subgroups, or supersoluble mutually permutable subgroups or supersoluble subgroups of prime indices. These results are presented in the following theorem.

Theorem 1.3. *Let A and B be supersoluble subgroups of G and $G = AB$. Then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$ in each of the following cases:*

(1) *A and B are subnormal, see* [11, Theorem 2]*;*

(2) *A and B are mutually permutable, see* [10, Theorem 2.1]*;*

(3) *the indices of A and B in G are prime, see* [12, Theorem C].

Here $\mathfrak{N}$ is the formation of all nilpotent groups.

A subgroup A of G is called *seminormal* in G, if there exists a subgroup B such that $G = AB$ and AX is a proper subgroup of G for every proper subgroup X of B, see [15]. The groups with some seminormal subgroups were investigated in works of many authors, see, for example, the literature in [12]. If the subgroups A and B of $G = AB$ are mutually permutable, then A and B are seminormal in G. The converse is false. For instance, $G = Z_7 \rtimes Z_6$ is the product of seminormal in G subgroups $A \simeq Z_6$ and $B \simeq Z_7 \rtimes Z_2$, but A and B are not mutually permutable. Here Z_n is a cyclic group of order n.

We introduce the following concept that unites subnormality and seminormality.

Definition. A subgroup A of G is called *semi-subnormal* in G, if A is subnormal in G or seminormal in G.

In this paper, we prove the following:

Theorem A. *Let A and B be semi-subnormal subgroups of G and $G = AB$. Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. If A and B belong to $\mathfrak{F}$, then $G^{\mathfrak{F}} \le (G')^{\mathfrak{N}}$.*

2. Preliminary results

In this section, we give some definitions and basic results which are essential in the sequel.

A group whose chief factors have prime orders is called *supersoluble*. The notation $H \le G$ means that H is a subgroup of G. If $H \le G$ and $H \ne G$, we write $H < G$. Denote by $Z(G)$, $F(G)$ and $\Phi(G)$ the centre, Fitting and Frattini subgroups of G, respectively, and by $\mathrm{O}_p(G)$ the greatest normal p-subgroup of G. Denote by $\pi(G)$ the set of all prime divisors of order of G. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$. If H is a subgroup of G, then $H_G = \bigcap_{x \in G} H^x$ is called *the core* of H in G.

A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. Let $\mathbb{P}$ be the set of all prime numbers. A *formation function* is a function f defined on $\mathbb{P}$ such that $f(p)$ is a, possibly empty, formation. A formation $\mathfrak{F}$ is said to be *local*, if there exists a formation function f such that $G \in \mathfrak{F}$ if and only if for any chief factor H/K of G and any $p \in \pi(H/K)$, one has $G/C_G(H/K) \in f(p)$. We write $\mathfrak{F} = LF(f)$, and f is a local definition of $\mathfrak{F}$. By [6, IV.3.7], among all possible local definitions of a local formation $\mathfrak{F}$, there exists a unique f such that f is integrated (i.e., $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and full (i.e., $f(p) = \mathfrak{N}_p f(p)$ for all $p \in \mathbb{P}$). Here $\mathfrak{N}_p$ is the formation of all p-groups. Such local definition f is said to be *canonical local definition* of $\mathfrak{F}$. By [6, IV.4.6], a formation is saturated if and only if it is local.

If G contains a maximal subgroup M with trivial core, then G is said to be *primitive*.

Lemma 2.1. *Let $\mathfrak{F}$ be a saturated formation. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Proof. Since $\mathfrak{F}$ is a saturated formation, it follows that $\Phi(G) = 1$ and G contains a unique minimal normal subgroup N. For some maximal subgroup M of G, we have $G = NM$, because $\Phi(G) = 1$. It is obvious that the core $M_G = 1$. Hence G is a primitive group. □

Lemma 2.2 ([7, Theorem II.3.2]). *If G is a soluble primitive group, then $F(G) = C_G(F(G)) = \mathrm{O}_p(G)$ is a unique minimal normal subgroup of G for some prime p.*

Lemma 2.3. (1) *If H is a semi-subnormal subgroup of G and $H \leq X \leq G$, then H is semi-subnormal in X.*

(2) *If H is a semi-subnormal subgroup of G and N is normal in G, then HN/N is semi-subnormal in G/N.*

Proof. If H is subnormal in G, then the statements (1)–(2) are true, see [8, Chapter 2]. If H is seminormal, then these statements were proved in [12, Lemma 4.1]. Thus the statements (1)–(2) are true. □

Lemma 2.4. *Let H be a maximal subgroup of G. If H is semi-subnormal in G, then the index of H in G is prime.*

Proof. If H is subnormal in G, then H is normal in G, and $|G : H|$ is prime by [9, Lemma 3.17 (6)]. If H is seminormal in G, then the conclusion of the lemma follows from [13, Theorem 1]. □

Lemma 2.5 ([9, Lemma 5.8]). *Let $\mathfrak{F}$ and $\mathfrak{H}$ be non-empty formations, and K be normal in G. Then:*

(1) $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$;

(2) $G \in \mathfrak{F}$ *if and only if* $G^{\mathfrak{F}} = 1$;

(3) *if* $\mathfrak{H} \subseteq \mathfrak{F}$, *then* $G^{\mathfrak{F}} \leq G^{\mathfrak{H}}$;

(4) $G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$.

Lemma 2.6 ([14, Lemma 2.16])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

3. Proof of Theorem A

We consider the case when the derived subgroup G' is nilpotent. Then G is soluble. Assume that $G \notin \mathfrak{F}$. If N is a non-trivial normal subgroup of G, then the subgroups AN/N and BN/N are semi-subnormal in G/N by Lemma 2.3 (2) and belong to $\mathfrak{F}$, because $\mathfrak{F}$ is a formation. Since

$$(G/N)' = G'N/N \simeq G'/G' \cap N,$$

it follows that the derived subgroup $(G/N)'$ is nilpotent. Therefore by induction, $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, we have that G is primitive by Lemma 2.1. Hence $\Phi(G) = 1$ and $N = C_G(N) = F(G) = \mathrm{O}_p(G)$ is a unique minimal normal subgroup of G by Lemma 2.2. Because G' is nilpotent, $N = G'$ and G/N is abelian.

Suppose that $AN = G$. Then $A \cap N = 1$ and A is a maximal subgroup of G. Since A is semi-subnormal in G, it follows by Lemma 2.4, the index of the subgroup A in G is prime. This means that $|N| = p$ and $G \in \mathfrak{F}$ by Lemma 2.6, a contradiction. Therefore, the assumption is wrong and $AN < G$. By Lemma 2.3 (1), A is semi-subnormal in AN. Since N is abelian, we have $N \in \mathfrak{F}$. Besides, $(AN)' \leq G'$, and hence $(AN)'$ is nilpotent. By induction, $AN \in \mathfrak{F}$. Similarly, we get that $BN < G$ and $BN \in \mathfrak{F}$. Thus $G = (AN)(BN)$ is the product of normal subgroups AN and BN such that each of them belongs to $\mathfrak{F}$.

Since AN is normal in G, it follows that $N \leq AN$, $\Phi(AN)=1$ and $F(AN)=N$. Hence $N = Y_1 \times Y_2 \times \cdots \times Y_k$, where Y_i is a minimal normal subgroup of AN for all i. Furthermore,

$$C_{AN}(N) = AN \cap C_G(N) = N.$$

By [9, Theorem 4.25], we have

$$N = F(AN) = \bigcap_i C_{AN}(Y_i).$$

Since $\mathfrak{F}$ is saturated, there exists the canonical local definition f. Hence $\mathfrak{F} = LF(f)$, $f(p) \subseteq \mathfrak{F}$ and $f(p) = \mathfrak{N}_p f(p)$. Then $AN/C_{AN}(Y_i) \in f(p)$ for any i. Because $f(p)$ is a formation, it follows that $AN/N \in f(p)$. Similarly, we get that $BN/N \in f(p)$.

We consider the direct product $AN/N \times BN/N = \{(aN, bN), a \in A, b \in B\}$. Let $\varphi : AN/N \times BN/N \to G/N = (AN/N)(BN/N)$ be a function from $AN/N \times BN/N$ to G/N and $\varphi(aN, bN) = (ab)N$. Since G/N is abelian, we have $AN/N \leq C_{G/N}(BN/N)$. It is clear that φ is an epimorphism. Then by [9, Theorem 2.3],

$$(AN/N \times BN/N)/\operatorname{Ker}\ \varphi \simeq \operatorname{Im}\ \varphi = G/N.$$

Since $f(p)$ is a formation, it follows that $G/N \in f(p)$. Because $N \in \mathfrak{N}_p$, we have $G \in \mathfrak{N}_p f(p) = f(p) \subseteq \mathfrak{F}$. Hence the assumption is wrong.

Let $(G')^{\mathfrak{N}} \neq 1$. We show that the quotient $G/(G')^{\mathfrak{N}}$ belongs to $\mathfrak{F}$. Since

$$(G/(G')^{\mathfrak{N}})' = G'(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} = G'/(G')^{\mathfrak{N}},$$

we have $(G/(G')^{\mathfrak{N}})'$ is nilpotent. The quotients

$$G/(G')^{\mathfrak{N}} = (A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}})(B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}),$$

$$A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} \simeq A/A \cap (G')^{\mathfrak{N}}, \qquad B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} \simeq B/B \cap (G')^{\mathfrak{N}},$$

hence the subgroups $A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ and $B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ belong to $\mathfrak{F}$, and by Lemma 2.3 (2), are semi-subnormal in $G/(G')^{\mathfrak{N}}$.

Arguing as above, we see that $G/(G')^{\mathfrak{N}}$ belongs to $\mathfrak{F}$. With this the theorem is proved. $\square$

Corollary 3.1. *Let $G = AB$, and $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. Suppose that A and B belong to $\mathfrak{F}$. If the derived subgroup G' is nilpotent, then $G \in \mathfrak{F}$ in each of the following cases:*

(1) *A and B are subnormal in G;*

(2) *A and B are seminormal in G;*

(3) *one of the subgroups A or B is seminormal in G, the other is subnormal in G;*

(4) *A and B are mutually permutable;*

(5) *the indices of A and B in G are prime.*

Since $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$, it follows that $G^{(\mathfrak{N}\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}} \leq G^{\mathfrak{U}}$ by Lemma 2.5 (3–4). Therefore, for $\mathfrak{F} = \mathfrak{U}$, Corollary 3.1 covers the above results of the papers [2], [10], [11], [12].

References

[1] M. J. Alejandre, A. Ballester-Bolinches and J. Cossey, Permutable products of supersoluble groups, *J. Algebra* **276** (2004), 453–461.

[2] M. Asaad and A. Shaalan, On the supersolvability of finite groups, *Arch. Math. (Basel)* **53** (1989), 318–326.

[3] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, Products of Finite Groups, *Walter De Gruyter*, *Berlin*, 2010.

[4] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, Mutually permutable products of finite groups. II, *J. Algebra* **218** (1999), 563–572.

[5] A. Ballester-Bolinches, M. C. Pedraza-Aguilera and M. D. Pérez-Ramos, On finite products of totally permutable groups, *Bull. Austral. Math. Soc.* **53** (1996), 441–445.

[6] K. Doerk and T. Hawkes, Finite Soluble Groups, *Walter De Gruyter*, *Berlin*, 1992.

[7] B. Huppert, Endliche Gruppen. I, *Springer-Verlag*, *Berlin – New York*, 1967.

[8] I. M. Isaacs, Finite Group Theory, *American Mathematical Society*, *Providence, RI*, 2008.

[9] V. S. Monakhov, Introduction to the Theory of Finite groups and their Classes, *Vysheĭshaja Shkola*, *Minsk*, 2006 (in *Russian*).

[10] V. S. Monakhov, On the supersoluble residual of mutually permutable products, *Probl. Fiz. Mat. Tekh.* **34** (2018), 69–70.

[11] V. S. Monakhov and I. K. Chirik, On the supersolvable residual of a product of subnormal supersolvable subgroups, *Sib. Math. J.* **58** (2017), 271–280.

[12] V. S. Monakhov and A. A. Trofimuk, Finite groups with two supersoluble subgroups, *J. Group Theory* **22** (2019), 297–312.

[13] V. V. Podgornaya, Seminormal subgroups and supersolvability of finite groups, *Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz.-Mat. Navuk* **4** (2000), 22–25 (in *Russian*).

[14] A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Algebra* **315** (2007), 192–209.

[15] X. Y. Su, Seminormal subgroups of finite groups, *J. Math. (Wuhan)* **8** (1988), 5–10 (in *Chinese*).

ALEXANDER TROFIMUK
DEPARTMENT OF MATHEMATICS
AND PROGRAMMING TECHNOLOGY
GOMEL FRANCISK SKORINA STATE UNIVERSITY
246019 GOMEL
BELARUS

E-mail: alexander.trofimuk@gmail.com

(Received April 19, 2019; revised September 5, 2019)