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Finite groups with subnormal non-cyclic subgroups

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Abstract. In this paper we consider finite groups G such that every non-cyclic maximal subgroup in its Sylow subgroups is subnormal in G. In particular, we prove that such solvable groups have an ordered Sylow tower.

1 Introduction

All groups considered in this paper will be finite. Our notation is standard and taken mainly from [7].

We say that G has a Sylow tower if there exists a normal series with each factor isomorphic to a Sylow subgroup of G.

Let G be a group of order $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where $p_1 > p_2 > \cdots > p_k$. We say that G has an ordered Sylow tower of supersolvable type if there exists a series

$$1 = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_{k-1} \subseteq G_k = G$$

of normal subgroups of G such that G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for each $i = 1, 2, \dots, k$.

Recall that a supersolvable group is a group which has a normal series with cyclic factors. If G is supersolvable, then G has an ordered Sylow tower of supersolvable type; see [7, VI.9.1]. The alternating group A_4 of degree 4 has a Sylow tower of non-supersolvable type.

By the Zassenhaus Theorem [7, IV.2.11], a group G with cyclic Sylow subgroups has a normal cyclic Hall subgroup such that the corresponding quotient group is also cyclic. Hence G is supersolvable.

In 1980 Srinivasan [9, Theorem 1] proved that if all maximal subgroups of the Sylow subgroups of G are normal in G, then G is supersolvable.

If the condition of normality is weakened to subnormality, then the group can be non-supersolvable. An example is the alternating group A_4 of degree 4. However, Srinivasan [9, Theorem 3] has proved that G has an ordered Sylow tower if all maximal subgroups of its Sylow subgroups are subnormal in G. The paper [9] found an echo in many papers; see [1–4].

Developing this theme we prove the following theorem.

Theorem 1.1. *Let G be a group. Assume that for all Sylow subgroups P of G and for all maximal subgroups M of P, if M is not cyclic, then M is subnormal in G. Then $S(G)$ has a Sylow tower and, if G is non-solvable, then*

$$G/S(G) \simeq \mathrm{PSL}(2, p),$$

p is prime, $p \equiv \pm 3 \pmod 8$.

Here $S(G)$ is a largest normal solvable subgroup of G.

2 Auxiliary results

Let G be a group and $\pi(G)$ be the set of primes dividing the order of G. Let p be a prime and G be a p-group. We also use the notation $\Omega_1(G) = \langle g \in G \mid g^p = 1 \rangle$. The center, the derived subgroup, the Frattini subgroup and the Fitting subgroup of G are denoted by $Z(G)$, G', $\Phi(G)$ and $F(G)$, respectively. By $O_p(G)$ and $O(G)$ we denote the greatest normal p-subgroup of G and the greatest normal subgroup of odd order of G, respectively. The notation $G = [A]B$ is used for a semidirect product with a normal subgroup A.

Lemma 2.1. *Let P be a non-cyclic p-group and assume that all the proper subgroups of P are cyclic. Then P is either elementary abelian of order p^2 or a quaternion group of order* 8.

Proof. Let $x \in Z(P)$ have order p. If there is a subgroup $\langle y \rangle$ of order p, different from $\langle x \rangle$, then $\langle x \rangle \langle y \rangle = \langle x \rangle \times \langle y \rangle$ is non-cyclic of order p^2 and so $P = \langle x \rangle \times \langle y \rangle$ is elementary abelian. If P has a unique subgroup of order p, then, by [7, III.8.2], P is a quaternion group of order 2^n, $n \geq 3$. Since all the subgroups of P are cyclic, it follows from [7, III.7.12] that P has the order 8. □

Lemma 2.2 ([5, Theorem 1.2]). *Let G be a non-abelian p-group of order p^{n+1} with cyclic subgroup $A = \langle a \rangle$ of index p. Then G is isomorphic to one of the following groups:*

(1) $M_{p^{n+1}} = \langle a, b \mid a^{p^n} = b^p = 1,\ a^b = a^{1+p^{n-1}} \rangle$, *where $n \geq 3$ if $p = 2$. In that case, $|G'| = p$, $Z(G) = \Phi(G)$, $|\Omega_1(G)| = p^2$.*

(2) $p = 2$ *and* $D_{2^{n+1}} = \langle a, b \mid a^{2^n} = b^2 = 1,\ bab = a^{-1} \rangle$, *the dihedral group. All elements in $G \backslash \langle a \rangle$ are involutions.*

(3) $p = 2$ *and* $Q_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ a^b = a^{-1} \rangle$, *the generalized quaternion group. The group G contains exactly one involution, all elements in $G \backslash \langle a \rangle$ have order* 4 *and, if $n > 2$, $G/Z(G)$ is dihedral.*

(4) *$p = 2$ and $SD_{2^{n+1}} = \langle a, b \mid a^{2^n} = b^2 = 1,\ bab = a^{-1+2^{n-1}} \rangle$, $n > 2$, the semidihedral group. We have $\Omega_1(G) = \langle a^2, b\rangle \simeq D_{2^n}$, $\langle a^2, ab\rangle \simeq Q_{2^n}$ so the maximal subgroups of G are characteristic in G, $G/Z(G)$ is dihedral.*

In cases (2)–(4)*, we have $|G : G'| = 4$, $|Z(G)| = 2$.*

For the remainder of this paper, we use the notation $M_{p^{n+1}}$, $D_{2^{n+1}}$, $Q_{2^{n+1}}$ and $SD_{2^{n+1}}$ for the groups listed in Lemma 2.2. By E_{p^n} we denote an elementary abelian group of order p^n.

Lemma 2.3. *Let P be a p-group and assume that P contains exactly one non-cyclic maximal subgroup. Then either $P = \langle a\rangle \times \langle b\rangle$ with $|a| > p$ and $|b| = p$, or $P \simeq M_{p^{n+1}}$, where $n \geq 2$ if $p > 2$, and $n \geq 3$ if $p = 2$.*

Proof. Assume that $|P| = p^{n+1}$ and that H is a non-cyclic maximal subgroup of P. Since P is not cyclic, there exists a maximal subgroup A of P with $A \neq H$. By hypothesis, A is cyclic. Let $A = \langle a\rangle$ and $b \in P \setminus A$. Then $P = \langle a\rangle\langle b\rangle$ and $|P/\Phi(P)| = p^2$; see [7, III.3.15]. Hence P has $1 + p$ maximal subgroups.

If P is abelian, then $P = \langle a\rangle \times \langle b\rangle$ for some $b \in P \setminus A$ and $|a| > p$ as $n \geq 2$. Now P has the following subgroups: p cyclic maximal subgroups $\langle a\rangle$, $\langle ab\rangle$, $\langle a^2 b\rangle, \ldots, \langle a^{p-1} b\rangle$ and one non-cyclic $H = \langle a^p\rangle \times \langle b\rangle$.

Let P be non-abelian and $p > 2$. By Lemma 2.2, P is isomorphic to $M_{p^{n+1}}$, and so contains one non-cyclic maximal subgroup $H = [\langle a^p\rangle]\langle b\rangle$, and p cyclic subgroups of index p; see [7, III.8.7]. Hence $M_{p^{n+1}}$ satisfies the requirements of the lemma if $p > 2$.

Let P be non-abelian and $p = 2$. By Lemma 2.2, there are four non-abelian 2-groups with cyclic maximal subgroup: $M_{2^{n+1}}$, $D_{2^{n+1}}$, $Q_{2^{n+1}}$ and $SD_{2^{n+1}}$.

The group $M_{2^{n+1}}$ contains two cyclic maximal subgroups $\langle a\rangle$ and $\langle ab\rangle$, and one non-cyclic $H = [\langle a^2\rangle]\langle b\rangle$. Hence $M_{2^{n+1}}$ satisfies the condition of the lemma.

The group $D_{2^{n+1}}$ contains two non-cyclic maximal subgroups $[\langle a^2\rangle]\langle b\rangle$ and $[\langle a^2\rangle]\langle ab\rangle$, hence $D_{2^{n+1}}$ does not satisfy the condition of the lemma.

All three maximal subgroups of Q_8 are cyclic. The group $Q_{2^{n+1}}$ contains two non-cyclic maximal subgroups $\langle a^2\rangle\langle b\rangle$ and $\langle a^2\rangle\langle ab\rangle$ if $n \geq 3$. Hence Q_8 and $Q_{2^{n+1}}$, $n \geq 3$, does not satisfy the condition of the lemma.

The group $SD_{2^{n+1}}$ contains two non-cyclic maximal subgroups D_{2^n} and Q_{2^n} and one cyclic maximal subgroup. Hence SD_{2^n} does not satisfy the condition of the lemma. The lemma is proved. □

A group is called p-closed if it has a normal Sylow p-subgroup. A group is called p-nilpotent if it has a normal p-complement.

Lemma 2.4. *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. Suppose that every non-cyclic maximal subgroup of P is* *subnormal in G. If G is not p-closed and is not p-nilpotent, then $p = 2$, 3 divides the order of G, and P is either the elementary abelian group of order 4 or the quaternion group of order 8.*

Proof. Note that for $p > 2$ the order of G is odd and coprime with $p^2 - 1$. Indeed, if q divides $|G|$ and q divides $p^2 - 1 = (p-1)(p+1)$, then $q > p$ and q divides $p + 1$. But this is possible only when $p = 2$ and $q = 3$; this is a contradiction.

If P is cyclic, then G is p-nilpotent by [7, IV.2.8].

If P has two non-cyclic maximal subgroups P_1 and P_2, then, by the hypothesis of the lemma, P_1 and P_2 are subnormal in G, hence $P = P_1 P_2$ is subnormal in G and, therefore, G is p-closed.

If P has exactly one non-cyclic maximal subgroup P_1, then either $P = \langle a \rangle \times \langle b \rangle$ with $|a| > p$, $|b| = p$, or $P \simeq M_{p^{n+1}}$, by Lemma 2.3. If $p > 2$, then G is p-nilpotent by [7, IV.5.10]. If $p = 2$, then G is 2-nilpotent by [7, IV.3.5, IV.2.7].

The remaining case is when all the maximal subgroups of P are cyclic. In this case, by Lemma 2.1, either $P \simeq E_{p^2}$ or P is a quaternion group of order 8.

If $p > 2$, then G is p-nilpotent by [7, IV.2.7].

Therefore $p = 2$ and P is either elementary abelian of order 4, or the quaternion group of order 8, and 3 divides $|G|$; see [7, IV.2.7, IV.5.10]. □

Let G be a group, $p \in \pi(G)$ and P be a Sylow p-subgroup of G. If every non-cyclic maximal subgroup of P is subnormal in G, then G is called an sr_p-group.

Lemma 2.5. *If G is an sr_p-group, H is a subgroup of G and N is normal in G, then H and G/N are sr_p-groups.*

Proof. Let P_1 be a Sylow subgroup of H and P be a Sylow subgroup of G such that $P_1 \subseteq P$. Suppose that M is a maximal subgroup of P_1, and assume that M is not cyclic. Then M is subnormal G and therefore also in H.

Let P be a Sylow p-subgroup of G and M/N be an arbitrary maximal subgroup of PN/N. Then there exists a subgroup K of P such that $KN/N = M/N$. Assume that M/N is not cyclic. Then K is not cyclic, and so K is subnormal in G. Therefore M/N is subnormal in G/N. The lemma is proved. □

3 Proof the theorem

By Lemma 2.5, the hypotheses of the theorem are inherited by all subgroups and quotients of G. Thus, to prove the theorem, we need to discuss solvable groups and groups with $S(G) = 1$.

Assume that $S(G) = 1$. Let P be a Sylow 2-subgroup of G. Since G is an sr_2-group and G does not contain subnormal 2-subgroups, it follows that all maximal subgroups of P are cyclic. If P is cyclic, then G has a normal 2-complement contrary to $S(G) = 1$. Hence, by Lemma 2.1, either $P \simeq E_{2^2}$, or $P \simeq Q_8$. If $P \simeq Q_8$, then $S(G) \neq 1$ by the Z^*-Theorem; see [6, Section 12.1.1]. Hence P is elementary abelian of order 4.

Let N be minimal normal in G. Then the subgroup N is simple and $|G : N|$ is odd. By [6, Theorem, p. 485], $N \simeq \mathrm{PSL}(2, p^n)$, p is prime, $p^n \equiv \pm 3 \pmod 8$. Since $p^n \equiv \pm 3 \pmod 8$, n is odd.

Let B be a Sylow p-subgroup of N. Then B is elementary abelian of order p^n. Since G is an sr_p-group and G does not contain subnormal p-subgroups, all the maximal subgroups of B are cyclic. By Lemma 2.1, B is either cyclic or $B \simeq E_{p^2}$. Since n is odd, we have $n = 1$.

As $C_G(N) \cap N = Z(N) = 1$, $C_G(N)$ is isomorphic to a subgroup of G/N. Hence $|C_G(N)|$ is odd. Since $C_G(N)$ is normal in G and $S(G) = 1$, we have $C_G(N) = 1$. Now G is isomorphic to a subgroup of $\mathrm{Aut}\, N$ containing $\mathrm{Inn}\, N$. It is well known that

$$\mathrm{Aut}\,\mathrm{PSL}(2, p) = \mathrm{PGL}(2, p), \quad |\mathrm{PGL}(2, p) : \mathrm{PSL}(2, p)| = 2.$$

Thus $G = N$.

Suppose now that G is solvable. By induction on $|G|$ we have

(1) For all $p \in \pi(G)$, G is not p-closed and G is not p-nilpotent.

The Frattini argument and (1) yield

(2) $\Phi(G) = Z(G) = 1$, $F(G) = C_G(F(G))$ and $F(G)$ has elementary abelian Sylow subgroups.

Let P be a Sylow 2-subgroup of G. Combining (1) and Lemma 2.4 gives

(3) P is either elementary abelian of order 4 or is a quaternion group of order 8.

We now show:

(4) For every odd $r \in \pi(G)$, a Sylow r-subgroup R of G is either cyclic or elementary abelian of order r^2, or $R = \langle a \rangle \times \langle b \rangle$, $|a| = r^2$, $|b| = r$, or $R \simeq M_{r^3}$.

To this end, let R be non-cyclic and $|R| \geq r^3$. Since R is non-normal in G, there is a unique non-cyclic maximal subgroup R_1 of R, by Lemma 2.1. By the condition of the theorem, it is subnormal in G, hence $R_1 \subseteq F(G)$. Now R_1 is elementary abelian. By Lemma 2.3, either $R = \langle a \rangle \times \langle b \rangle$, $|a| > r$, $|b| = r$, or $R \simeq M_{r^{n+1}}$. If $R = \langle a \rangle \times \langle b \rangle$, then we have $R_1 = \langle a^r \rangle \times \langle b \rangle$ and $|a| = r^2$. If $R \simeq M_{r^{n+1}}$, then $R_1 = [\langle a^r \rangle]\langle b \rangle$ and $|a| = r^2$.

(5) $H = G_{2'}$ has an ordered Sylow tower of supersolvable type.

By (4), every Sylow subgroup of H is metacyclic. Now G has an ordered Sylow tower of supersolvable type; see [6, 7.6.3] and [8, Corollary of Theorem 2].

(6) G is a $\{2, 3\}$-group.

Choose $r \in \pi(H)$ maximal. The Sylow r-subgroup R of H is a Sylow r-subgroup of G and R is normal in H, by (5). It is clear that $|G : N_G(R)| \in \{4, 8\}$. If $|G : N_G(R)| = 8$, then $P \simeq Q_8$. Thus the center of $G/O(G)$ contains the involution of $PO(G)/O(G)$ by the Z^*-Theorem; see also [6, 12.1.1]. Let $\langle i \rangle \leq P$ have order 2. Then $\langle i \rangle O(G)/O(G) \subseteq Z(G/O(G))$, hence

$$(\langle i \rangle O(G)/O(G))(H/O(G)) = \langle i \rangle H/O(G)$$

is a subgroup of $G/O(G)$. But now H is normal in $\langle i \rangle H$ and $N_G(R) \supseteq \langle i \rangle H$. Therefore $|G : N_G(R)| \neq 8$, which is a contradiction. Hence $|G : N_G(R)| = 4$ and $r = 3$ by Sylow's theorem.

(7) End of proof.

Suppose that $|F(G)|$ is even. Then the Sylow 2-subgroup F_2 of $F(G)$ is elementary abelian and nontrivial. Since $F_2 \subseteq P$ and $F_2 \neq P$, by (1), it follows that $|F_2| = 2$ and $F_2 \subseteq Z(G)$, this contradicts (2). Therefore the assumption is false, and $F(G) = O_3(G)$. By (4), we find that $|F(G)| = 3$ or 9. By (2), $F(G)$ coincides with its centralizer. Hence $G/F(G)$ is isomorphic to a subgroup of $\operatorname{Aut} F(G)$. If $|F(G)| = 3$, then $|G/F(G)| = 2$ and G is supersolvable. Thus $|F(G)| = 3^2$. Since $\Phi(G) = 1$, there is a subgroup M of G such that $G = [F(G)]M$. Now, for a Sylow 3-subgroup R, we have $R = [F(G)](R \cap M)$. But this is impossible in the metacyclic group R of order 3^3.

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