UDC 539.12

Maxwell equations in Lobachevsky space, and modeling the medium with reflecting properties

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Abstract

Lobachevsky geometry simulates a medium with special constitutive relations, $D^i = \varepsilon_0 \varepsilon^{ik} E^k, B^i = \mu_0 \mu^{ik} H^k$, where two matrices coincide: $\varepsilon^{ik}(x) = \mu^{ik}(x)$. The situation is specified in quasi-Cartesian coordinates (x, y, z) in Lobachevsky space, they are appropriate for modeling a medium nonuniform along the axis z. Exact solutions of the Maxwell equations in complex form of Majorana – Oppenheimer have been constructed. The problem reduces to a second order differential equation for a certain primary function which can be associated with the one-dimensional Schrödinger problem for a particle in external potential field $U(z) = U_0 e^{2z}$. In the frames of the quantum mechanics, the Lobachevsky geometry acts as an effective potential barrier with reflection coefficient $R = 1$; in electrodynamic context results are similar: this geometry simulates a medium that effectively acts as an ideal mirror distributed in space. Penetration of the electromagnetic field into the effective medium along the axis z, depends on the parameters of an electromagnetic waves $\omega, k_1^2 + k_2^2$, and the curvature radius ρ of the used Lobachevsky model. The generalized quasi-plane wave solutions $f(t, x, y, z) = E + iB$ and the relevant system of equations are transformed the real form, which permit us to relate geometry characteristics with expressions for effective tensors of electric and magnetic permittivities.

Keywords: Maxwell equations, Majorana – Oppenheimer formalism, Lobachevsky geometry, exact solutions, effective constitutive relations

Уравнения Максвелла в пространстве Лобачевского, моделирование среды со специальными свойствами

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Аннотация

Геометрия Лобачевского моделирует среду с материальными уравнениями специального вида: $D^i = \varepsilon_0 \varepsilon^{ik} E^k, B^i = \mu_0 \mu^{ik} H^k$, где два тензора совпадают: $\varepsilon^{ik}(x) = \mu^{ik}(x)$. В пространстве Лобачевского используются квазидекартовые координаты (x, y, z), они моделируют среду, неоднородную вдоль оси z. В этих координатах построены точные решения уравнений Максвелла в комплексной форме Майораны – Оппенгеймера. Задача сводится к дифференциальному уравнению второго порядка для некоторой основной функции, это уравнение может быть связано с одномерной задачей Шредингера для частицы во внешнем потенциальном поле $U(z) = U_0 e^{2z}$. В квантовой механике геометрия Лобачевского действует как эффективный потенциальный барьер с коэффициентом отражения $R = 1$; в электродинамическом контексте эта геометрия действует как распределенное в пространстве идеальное зеркало. Проникновение электромагнитного поля в эффективную среду вдоль оси z зависит от характеристик электромагнитной волны $\omega, k_1^2 + k_2^2$ и радиуса кривизны ρ пространства Лобачевского. Построенные обобщенные волновые решения $f(t, x, y, z) = E + iB$ и соответствующая система уравнений преобразуются в действительную форму, что позволяет связать геометрические характеристики с выражениями для эффективных тензоров электрической и магнитной проницаемостей.

Ключевые слова: уравнения Максвелла, формализм Майораны – Оппенгеймера, геометрия Лобачевского, точные решения, моделирование материальных сред

Introduction

To treat Maxwell equations we make use of complex representation of them according to the known approach by Majorana – Oppenheimer [1–11], also see [12, 13] and references therein for extending this approach to curved space-time models.

The situation is specified in quasi-Cartesian coordinates (x, y, z) in Lobachevsky space, they are appropriate for modeling a medium nonuniform along the axis z. Exact solutions of the covariant Maxwell equations in complex $E + iB$ form of Majorana – Oppenheimer have been constructed. The problem reduces to a second order differential equation for a certain primary function which can be associated with the one-dimensional Schrödinger problem for a particle in external potential field $U(z) = U_0 e^{2z}$. In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient $R = 1$; in electrodynamic context results are similar: the Lobachevsky geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the electromagnetic field into the effective medium along the axis z, depends on the parameters of the electromagnetic waves $\omega, k_1^2 + k_2^2$, and the curvature radius ρ of the used Lobachevsky space. These generalized quasi-plane solutions $f(t, x, y, z) = E + iB$ and the relevant system of equations are transformed the real form, which permit us to relate geometry characteristics with expressions for effective tensors of electric and magnetic permittivities.

1. Cartesian coordinates in Lobachevsky space

We will apply the coordinate system in Lobachevsky space H_3

$$dS^2 = dt^2 - e^{-2z}(dx^2 + dy^2) - dz^2 \,, \; dV = e^{-2z} dxdydz \,. \qquad (1)$$

It is helpful to have at hand some details of the parametrization of the model H_3 by coordinates (x, y, z). It is known that this model can be identified with a branch of hyperboloid in 4-dimension flat space

$$u_0^2 - u_1^2 - u_2^2 - u_3^2 = \rho^2, \qquad u_0 = +\sqrt{\rho^2 + u^2}.$$

Coordinates (x, y, z) are referred to u_a by relations

$$u_0 = \frac{1}{2}[(e^z + e^{-z}) + (x^2 + y^2)e^{-z}], \quad u_1 = xe^{-z} ,$$

$$u_2 = ye^{-z}, \quad u_3 = \frac{1}{2}[(e^z - e^{-z}) + (x^2 + y^2)e^{-z}].$$

We will employ the Poincaré realization for Lobachevsky space as the inside part of the 3-sphere

$$q_i = \frac{u_i}{u_0} = \frac{u_i}{\sqrt{\rho^2 + u_1^2 + u_2^2 + u_3^2}}, \qquad q_i q_i < +1 .$$

Quasi-Cartesian coordinates (x, y, z) are referred to q_i as follows

$$q_1 = \frac{2x}{x^2 + y^2 + e^{2z} + 1}, q_2 = \frac{2y}{x^2 + y^2 + e^{2z} + 1}, q_3 = \frac{x^2 + y^2 + e^{2z} - 1}{z^2 + y^2 + e^{2z} + 1}; \tag{2}$$

inverses to (2) relations are

$$x = \frac{q_1}{1 - q_3}, \quad y = \frac{q_2}{1 - q_3}, \quad e^z = \frac{\sqrt{1 - q^2}}{1 - q_3} . \tag{3}$$

In particular, note that on the axis $q_1 = 0, q_2 = 0, q \in (-1, +1)$ relations (3) assume the following parametrization of the axis z :

$$x = 0, \quad y = 0, \quad e^z = \sqrt{\frac{1 + q_3}{1 - q_3}},$$

so that

$$q_3 \to +1, e^z \to +\infty, z \to +\infty ; \qquad q_3 \to -1, e^z \to +0, z \to -\infty .$$

Solutions of the Maxwell equations, constructed in the following, can be of interest for description of electromagnetic waves in special media, because the Lobachevsky geometry simulates effectively a special medium [12, 13], inhomogeneous along the axis z . Effective electric permittivity tensor $m^{ik}(x)$ is given by

$$m^{ik}(x) = -\sqrt{-g}\; g^{00}(x) g^{ik}(x) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2z} \end{vmatrix},$$

whereas the effective magnetic permittivity tensor is

$$(\mu^{-1})^{ik}(x) = \sqrt{-g} \begin{vmatrix} g^{22}g^{33} & 0 & 0 \\ 0 & g^{33}g^{11} & 0 \\ 0 & 0 & g^{11}g^{22} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2z} \end{vmatrix}.$$

The constitutive relations read

$$D^i = m_0 m^{ik} E_k , \quad B_i = \mu_0 \mu^{ik} H^k ;$$

two tensors coincide, $m^{ik}(x) = (\mu^{-1})^{ik}(x)$.

2. Maxwell equations in complex form, separation of the variables

In the coordinates (1), we will use the tetrad

$$e_{(a)}^{\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & e^z & 0 & 0 \\ 0 & 0 & e^z & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad e_{(a)\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-z} & 0 & 0 \\ 0 & 0 & -e^{-z} & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix};$$

in this tetrad, the matrix equation (see notations in [12, 13]) has the form

$$(-i\partial_t + \alpha^1 e^z \partial_x + \alpha^2 e^z \partial_y + \alpha^3 \partial_z - \alpha^1 s_2 + \alpha^2 s_1) \begin{vmatrix} 0 \\ E + iB \end{vmatrix} = 0 \,. \tag{4}$$

Let us apply the substitution

$$\begin{vmatrix} 0 \\ E + iB \end{vmatrix} = e^{-i\omega t} \; e^{ik_1 x} \; e^{ik_2 y} \begin{vmatrix} 0 \\ f(z) \end{vmatrix}, \quad = e^{i(k_1 x + k_2 y - \omega t)} = e^{i\varphi}.$$

Equation (4) gives

$$(-\omega + \alpha^1 e^z ik_1 + \alpha^2 e^z ik_2 + \alpha^3 \frac{d}{dz} - \alpha^1 s_2 + \alpha^2 s_1) \begin{vmatrix} 0 \\ f_1(z) \\ f_2(z) \\ f_3(z) \end{vmatrix} = 0.$$

After calculation with the use of explicit expressions for all involved matrices (see [12, 13]), we derive the first order system for functions $f_1(z), f_2(z), f_3(z)$:

$$ik_1 \, e^z f_1 + ik_2 \, e^z \; f_2 + (\frac{d}{dz} - 2) f_3 = 0, \quad -\omega f_1 - (\frac{d}{dz} - 1) f_2 + ik_2 \, e^z \; f_3 = 0,$$

$$-\omega f_2 + (\frac{d}{dz} - 1) f_1 - ik_1 e^z f_3 = 0, \quad -\omega f_3 - e^z ik_2 f_1 + ik_1 e^z f_2 = 0.$$

Allowing for three last equations in the first one, we get the identity $0 = 0$. So, there exist only three independent equations (we will simplify notations, $k_1 = a, k_2 = b$):

$$\omega f_3 = -ibe^z f_1 + iae^z f_2,$$

$$\omega f_1 = -(\frac{d}{dz} - 1) f_2 + ibe^z \; f_3, \quad \omega f_2 = +(\frac{d}{dz} - 1) f_1 - iae^z f_3. \tag{5}$$

With substitutions $f_1 = e^z F_1(z), \; f_2 = e^z F_2(z)$, from eqs. (5) we get

$$\omega f_3 = -ibe^{2z} F_1 + iae^{2z} F_2, \quad \omega F_1 = -\frac{d}{dz} F_2 + ibf_3, \quad \omega F_2 = \frac{d}{dz} F_1 - iaf_3. \tag{6}$$

There exists a particular case readily treatable, when $a = 0, b = 0, \; f_3 = 0$:

$$\omega F_1 = -\frac{d}{dz} F_2, \;\; \omega F_2 = +\frac{d}{dz} F_1 \quad \Rightarrow \quad F_1(z) = e^{\pm i\omega z}, \, F_2 = \pm ie^{\pm i\omega z},$$

which leads to the following plane wave solutions

$$\Phi^{\pm} = \begin{vmatrix} 0 \\ E + iB \end{vmatrix} = e^{-i\omega t} e^z \begin{vmatrix} 0 \\ e^{\pm i\omega z} \\ \pm i \; e^{\pm i\omega z} \\ 0 \end{vmatrix},$$

whence we get

$$E_1^+ + iB_1^+ = \cos(\omega t - \omega z) - i \sin(\omega t - \omega z),$$

$$E_2^+ + iB_2^+ = \sin(\omega t - \omega z) + i \cos(\omega t - \omega z),$$

and

$$E_1^- + iB_1^- = \cos(\omega t + \omega z) - i \sin(\omega t + \omega z),$$

$$E_2^- + iB_2^- = -\sin(\omega t + \omega z) - i \cos(\omega t + \omega z).$$

Let us present this solution in the real form

$$E_1^+ = \cos(\omega t - \omega z), \quad E_2^+ = \sin(\omega t - \omega z), E_3^+ = 0,$$
$$B_1^+ = -\sin(\omega t - \omega z), \quad B_2^+ = \cos(\omega t - \omega z), B_3^+ = 0,$$

and

$$E_1^- = \cos(\omega t + \omega z), \quad E_2^- = -\sin(\omega t + \omega z), E_3^- = 0,$$
$$B_1^- = -\sin(\omega t + \omega z), \quad B_2^- = -\cos(\omega t + \omega z), B_3^- = 0.$$

In turn, from complex-valued identities (in this case, we have $\varphi = -\omega t$)

$$E + iB = e^{i\varphi} f(z) = e^{i\varphi}(F(z) + iG(z)) = (\cos\varphi + i\sin\varphi)(F(z) + iG(z)),$$
$$F^* = F, \quad G^* = G, \quad \varphi = k_1 x + k_2 y - \omega t,$$

we derive expressions for real vectors E and B:

$$E = \cos\varphi F(z) - \sin\varphi G(z), \quad B = \sin\varphi F(z) + \cos\varphi G(z), \quad \varphi = -\omega t.$$

Let us turn back to the general system (6); with the help of the first equation we eliminate the variable f_3, so producing the system of linked equations for F_1 and F_2:

$$\left(\frac{d}{dz} + \frac{abe^{2z}}{\omega}\right)F_2 = \frac{b^2 e^{2z} - \omega^2}{\omega} F_1,$$
$$\left(\frac{d}{dz} - \frac{abe^{2z}}{\omega}\right)F_1 = \frac{\omega^2 - a^2 e^{2z}}{\omega} F_2. \qquad (7)$$

In the new variable Z, $e^z = \sqrt{\omega}\, Z$, two last equations are written as

$$Z\left(\frac{d}{dZ} + abZ\right)F_2 = +(b^2 Z^2 - \omega)F_1,$$
$$Z\left(\frac{d}{dZ} - abZ\right)F_1 = -(a^2 Z^2 - \omega)F_2. \qquad (8)$$

This system can be solved straightforwardly in terms of the Heun confluent functions. Indeed, from (8) it follows a second order differential equation for F_1

$$\frac{d^2 F_1}{dZ^2} - \frac{a^2 Z^2 + \omega}{Z(a^2 Z^2 - \omega)} \frac{dF_1}{dZ} + \left(\frac{\omega^2}{Z^2} + \frac{2ab\omega}{a^2 Z^2 - \omega} - (a^2 + b^2)\omega\right)F_1 = 0,$$

where we note the presence of an additional singular point $Z = \pm\sqrt{\omega}/a$. In the new variable $y = a^2 Z^2/\omega$, we arrive at the equation

$$\frac{d^2 F_1}{dy^2} + \left(\frac{1}{y} - \frac{1}{y-1}\right)\frac{dF_1}{dy} + \left(\frac{\omega^2}{4y^2} - \frac{2ab\omega + (a^2 + b^2)\omega^2}{4a^2 y} + \frac{b\omega}{2a(y-1)}\right)F_1 = 0.$$

With the use of the substitution $F_1 = y^c g_1(y), c = \pm i\omega/2$, further we derive

$$\frac{d^2 g_1}{dy^2} + \left(\frac{2c+1}{y} - \frac{1}{y-1}\right)\frac{dg_1}{dy} + \left(\frac{2c - \omega^2/2 - b\omega/a - b^2\omega^2/(2a^2)}{2, y} + \frac{-2c + b\omega/a}{2(y-1)}\right)g_1 = 0,$$

which can be identified with the confluent Heun equation. Below we will develop a method that makes possible to construct solutions of the system (7) in terms of more simple Bessel functions.

3. Solutions in terms of the Bessel functions

Let us perform a linear transformation over the system (7):

$$F_1 = \alpha\, G_1 + \beta\, G_2, \qquad F_2 = m\, G_1 + n\, G_2;$$
$$G_1 = n\, F_1 - \beta\, F_2, \qquad G_2 = -m\, F_1 + \alpha\, F_2; \qquad (9)$$

suppose the constraint $\alpha n - \beta m = 1$. Combining equations from (7), we get

$$n\,Z\,(\frac{d}{dZ} - ab\,Z)\,F_1 - \beta\,Z\,(\frac{d}{dZ} + ab\,Z)\,F_2 = -n\,(a^2Z^2 - \omega)\,F_2 - \beta\,(b^2Z^2 - \omega)\,F_1,$$

$$-m\,Z\,(\frac{d}{dZ} - ab\,Z)\,F_1 + \alpha\,Z\,(\frac{d}{dZ} + ab\,Z)\,F_2 = m\,(a^2Z^2 - \omega)\,F_2 + \alpha\,(b^2Z^2 - \omega)\,F_1,$$

whence it follows

$$Z\,\frac{d}{dZ}\,G_1 - Z^2\,ab\,(nF_1 + \beta F_2) = -Z^2\,(na^2F_2 + \beta b^2F_1) + \omega\,(nF_2 + \beta F_1),$$

$$Z\,\frac{d}{dZ}\,G_2 + Z^2\,ab\,(mF_1 + \alpha F_2) = Z^2\,(ma^2F_2 + \alpha b^2F_1) - \omega\,(mF_2 + \alpha F_1). \qquad (10)$$

Taking into account (9), we reduce eqs. (10) to other form

$$\left[Z\,\frac{d}{dZ} - Z^2ab(n\alpha + \beta m) + Z^2(a^2mn + b^2\alpha\beta) - \omega(nm + \alpha\beta)\right]G_1 =$$

$$= \left[-Z^2(an - b\beta)^2 + \omega(n^2 + \beta^2)\right]G_2,$$

$$\left[Z\,\frac{d}{dZ} + Z^2ab\,(m\beta + n\alpha) - Z^2(a^2mn + b^2\alpha\beta) + \omega(nm + \alpha\beta)\right]G_2 =$$

$$= \left[Z^2(am - b\alpha)^2 - \omega(m^2 + \alpha^2)\right]G_1.$$

Let us impose additional restrictions:

the first one is

$$an - b\beta = 0 \qquad \Rightarrow \qquad \frac{\beta}{n} = \frac{a}{b},$$

$$[Z\,\frac{d}{dZ} - Z^2ab(n\alpha + \beta m) + Z^2(a^2mn + b^2\alpha\beta) - \omega(nm + \alpha\beta)]G_1 = +\omega(n^2 + \beta^2)G_2,$$

$$[Z\,\frac{d}{dZ} + Z^2ab(m\beta + n\alpha) - Z^2(a^2mn + b^2\alpha\beta) + \omega(nm + \alpha\beta)]G_2 =$$

$$= [Z^2(am - b\alpha)^2 - \omega\,(m^2 + \alpha^2)]G_1; \qquad (11)$$

the second one is

$$am - b\alpha = 0 \qquad \Rightarrow \qquad \frac{\alpha}{m} = \frac{}{b},$$

$$[Z\,\frac{d}{dZ} - Z^2ab(n\alpha + \beta m) + Z^2(a^2mn + b^2\alpha\beta) - \omega(nm + \alpha\beta)]G_1 =$$

$$= [-Z^2(an - b\beta)^2 + \omega(n^2 + \beta^2)]G_2,$$

$$[Z\,\frac{d}{dZ} + Z^2ab(m\beta + n\alpha) - Z^2(a^2mn + b^2\alpha\beta) + \omega(nm + \alpha\beta)]G_2 = -\omega(m^2 + \alpha^2)G_1.$$

These two possibilities are equivalent to each other, for definiteness we will use the variant (11). It can be presented in more symmetrical form

$$F_1 = \alpha\,G_1 + \beta\,G_2 = +\frac{b}{\sqrt{a^2 + b^2}}\,G_1 + \frac{a}{\sqrt{a^2 + b^2}}\,G_2,$$

$$F_2 = m\,G_1 + n\,G_2 = -\frac{a}{\sqrt{a^2 + b^2}}\,G_1 + \frac{b}{\sqrt{a^2 + b^2}}\,G_2; \qquad (12)$$

at this eqs. (6) lead to

$$[Z\frac{d}{dZ} - Z^2 ab\frac{b^2-a^2}{b^2+a^2} + Z^2 ab\frac{b^2-a^2}{b^2+a^2} - \omega(-\frac{ab}{a^2+b^2} + \frac{ab}{a^2+b^2})]G_1 =$$
$$= +\omega(\frac{b^2}{a^2+b^2} + \frac{a^2}{a^2+b^2})G_2,$$
$$[Z\frac{d}{dZ} + Z^2 ab\frac{b^2-a^2}{a^2+b^2} - Z^2 ab\frac{b^2-a^2}{a^2+b^2} + \omega(-\frac{ab}{a^2+b^2} + \frac{ab}{a^2+b^2})]G_2 =$$
$$= [Z^2(-\frac{a^2}{\sqrt{a^2+b^2}} - \frac{b^2}{\sqrt{a^2+b^2}})^2 - \omega(\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2})]G_1,$$

whence it follows

$$Z\frac{d}{dZ}G_1 = \omega G_2, \qquad Z\frac{d}{dZ}G_2 = [Z^2(a^2+b^2) - \omega]G_1. \tag{13}$$

From (13) we derive a second order equation for G_1:

$$(Z^2\frac{d^2}{dZ^2} + Z\frac{d}{dZ} + \omega^2 - \omega(a^2+b^2)Z^2)G_1 = 0. \tag{14}$$

It is convenient to translate this equation to the initial variable z, then it reads

$$e^z = \sqrt{\omega}\,Z, \qquad (\frac{d^2}{dz^2} + \omega^2 - (a^2+b^2)e^{2z})G_1 = 0. \tag{15}$$

It can be associated with the Schrödinger equation

$$(\frac{d^2}{dz^2} + m - U(z))\varphi(z) = 0 \tag{16}$$

with the potential function $U(z) = (a^2+b^2)e^{2z}$, the corresponding effective force acts on the left, $F_z = -2(a^2+b^2)e^{2z}$. The situation described by eq. (15) can be illustrated by Fig.1.

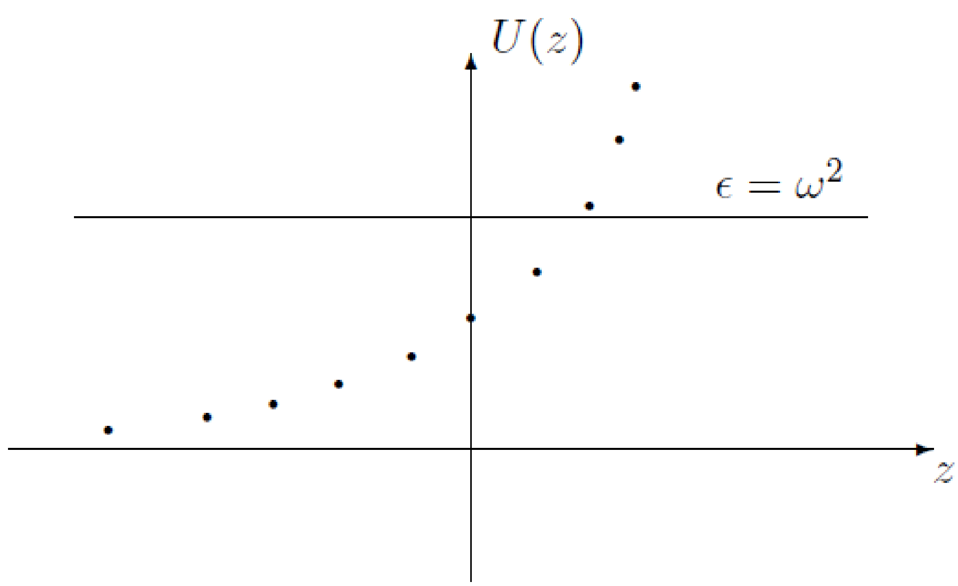

Figure 1 – Effective potential curve

Therefore, we should expect the properties of the electromagnetic solutions similar to those existing in the relevant quantum-mechanical problem. Note that when $a = k_1 = 0, b = k_2 = 0$, this force vanishes. In accordance with (16), an equation below $\omega^2 = U(z)$ $\omega^2 = (a^2+b^2)e^{2z_0}$ determines a critical point z_0 in which behavior of the function $G_1(x)$ must change dramatically. To such a point z_0 there corresponds $z_0 \Rightarrow x_0 = i\sqrt{a^2+b^2}e^{z_0} = i\omega$. Expression for the turning point z_0 is given by the formula

$$z_0 = \rho \ln \frac{\omega}{\rho\sqrt{k_1^2+k_2^2}};$$

the last relation is written in the usual units; the ρ is a curvature radius of the Lobachevsky space, it is a free parameter of the model description.

The primary variable $G_1(x)$ determine all remaining ones. Let us turn back to eq. (14); in the variable $x = i\sqrt{\omega(a^2+b^2)}Z = i\sqrt{a^2+b^2}e^z$ it takes the Bessel form

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 + \frac{\omega^2}{x^2}\right)G_1 = 0 .$$

The first order system (13), being transformed to the variable x, reads

$$x\frac{d}{dx}G_1 = \omega G_2, \qquad x\frac{d}{dx}G_2 = -\frac{\omega^2+x^2}{\omega}G_1.$$

The second function is determined by relation

$$G_2 = \frac{1}{\omega}x\frac{d}{dx}G_2 = \frac{1}{\omega}\frac{d}{dz}G_1.$$

In turn, taking into account the transformation (12), we get (see (6))

$$f_3 = \frac{e^{2z}}{\omega}(-ib\,F_1 + ia\,F_2) = \frac{\sqrt{a^2+b^2}}{i\,\omega}\,e^{2z}G_1(z)\,.$$

Let us write down the final expressions for obtained solutions:

$$E(z) + iB(z) = (\cos\varphi + i\sin\varphi)\ f(z), \quad \varphi = ax + by - i\omega t,$$

where

$$f_1(z) = e^z F_1(z) = e^z\left(+\frac{b}{\sqrt{a^2+b^2}}\,G_1 + \frac{a}{\sqrt{a^2+b^2}}\,G_2\right),$$

$$f_2(z) = e^z F_2(z) e^z\left(-\frac{a}{\sqrt{a^2+b^2}}\,G_1 + \frac{b}{\sqrt{a^2+b^2}}\,G_2\right),$$

$$f_3(z) = -i\frac{\sqrt{a^2+b^2}}{\omega}\,e^{2z}G_1(z)$$

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 + \frac{\omega^2}{x^2}\right)G_1(x) = 0, \quad G_2(z) = \frac{1}{\omega}\frac{d}{dz}G_1(z), \quad x = i\sqrt{a^2+b^2}\,e^z.$$

Conclusion

In the frames of the quantum mechanics, the Lobachevsky geometry acts as an effective potential barrier with reflection coefficient $R=1$; in electrodynamic context results are similar: this geometry simulates a medium that effectively acts as an ideal mirror distributed in space. Penetration of the electromagnetic field into the effective medium along the axis z, depends on the parameters of an electromagnetic waves $\omega, k_1^2+k_2^2$, and the curvature radius ρ of the used Lobachevsky model. The generalized quasi-plane wave solutions $f(t,x,y,z) = E + iB$ and the relevant system of equations are transformed the real form, which permit us to relate geometry characteristics with expressions for effective tensors of electric and magnetic permittivities.

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