СЕКЦИЯ 1. ОДАРЕННОСТЬ В ОБЛАСТИ МАТЕМАТИКИ КАК ПРЕДМЕТ НАУЧНЫХ ИССЛЕДОВАНИЙ

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DUALITY IN ACTION: THE LEXELL CIRCLE

Introduction: Methods and goals. School mathematics usually treats a given branch of mathematics on the basis of a single fixed system. Examples are the algebraic properties of addition and multiplication in arithmetic, or the Euclidean system in the field of geometry.

The limited perspective makes it difficult for the student to grasp the deeper meaning of a concept, or to understand different views and systems inside and outside mathematics. Besides, it alienates the student from mathematics which appears to him as an essentially closed discipline with only small details left for further development.

From this point of view, it is very useful to study examples and systems that inspire the student to step out f the box, rethink the meaning of a concept or theorem from a new aspect. This article shows how to achieve these goals in two different ways, namely, via Comparative geometry between the plane and the sphere, and the principle of dualization within the same geometric system.

First, we consider an example in Euclidean geometry of the plane. Next, we translate the example from the plane to the sphere. Finally we dualize our results within synthetic spherical geometry.

Theorem 1: Construction on the plane. Given a triangle on the plane, and one of its excircles that touches one side and the extension of the two other sides. With the two extended sides fixed, draw another third side which is also tangential to the excircle, and get a new triangle. Prove that the sum of sides is the same in the two triangles (Figure 1).

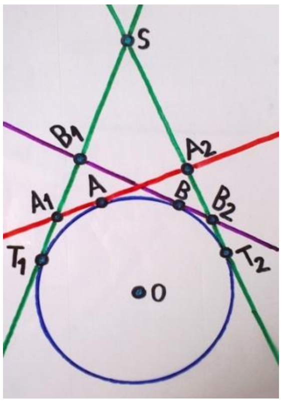

Figure 1

Triangle SA_1A_2 has the excircle with center O, touching the sides at points A, T_1, T_2. The construction shows that $AA_1 = A_1T_1$, $AA_2 = A_2T_2$ which gives $SA_1 + A_1A + AA_2 = ST_1 + ST_2$.

The same reasoning goes for triangle SB_1B_2, because $BB_1 = B_1T_1$, $BB_2 = B_2T_2$, therefore $SB_1 + B_1B + BB_2 = SA_1 + A_1A + AA_2 = ST_1 + ST_2$. But $ST_1 + ST_2$ does not depend on the location of the third side. This proves that the sum of sides is a constant $ST_1 + ST_2$ in all possible cases.

Theorem 2: Construction on the sphere. Theorem 1 says: "Given a triangle on the plane, and one of its excircles that touches one side and the extension of the two other sides. With the two extended sides fixed, draw another third side which is also tangential to the excircle, and get a new triangle. Prove that the sum of sides is the same in the two triangles".

When translating it onto the sphere, the only problem is to interpret the concept of "excircle" on the plane. A reasonable proposal is to replace the "excircle" by the inscribed circle in one of the supplementary triangles of the spherical triangle.

By doing so, the whole construction can be completed and proved in the same manner as we did on the plane. The only difference is the existence of point *S'* which is the opposite point of point *S* (Figure 2). This gives that the theorem is valid both on the plane and one the sphere. It is worth remarking that the same construction works in hyperbolic geometry as well. So this theorem belongs to the realm of absolute geometry.

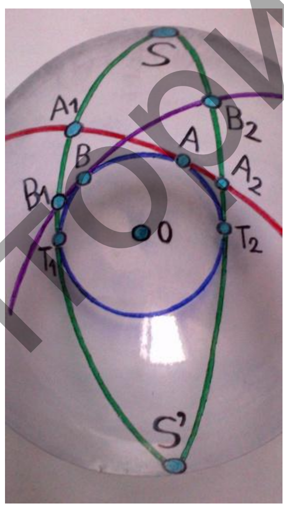

Figure 2

The principle of duality. The princilpe of duality states the following: Given a theorem about points, straight lines and circles, and metric relations of distance between two points and angle of two straight lines. Replace points by straight lines, straight lines by points, circles by another circles, distance by angle and angle by distance. In addition, preserve the relation of incidence: If point *A* is incident to straight line *b*, then point *B* is incident to straight line *a*. (This means that if point *A* is on straight line *b* in the original construction, the dual of point *A* which is straight line *a* goes through the dual of straight line *b* which is point *B*.)

If these conditions are all fulfilled, we get a dual theorem which is also correct. From this it follows that an inscribed circle touching the three sides of a triangle is dualized by a circumscribed circle through the three vertices of another triangle.

The principle of duality can be applied with full rigor on the plane if we complete the plane with ideal points of parallel straight lines, and also on the sphere if we take a pair of opposite points as a single point. In the present article we focus on the spherical case.

The dual construction of Theorem 1 on the sphere. The original construction the sphere is: “Given a triangle on the sphere, and an inscribed circle in one of its supplementary triangles. With the two extended sides of the original triangle fixed, draw another third side between the original triangle and the given supplementary triangle so that the new common side is also tangential to the circle. Prove that the sum of sides is the same in the two triangles” (Figure 2).

When dualizing the theorem, it is easy to replace the triangle with a trilateral, that is, another triangle, and the inscribed circle with a circumscribed circle through three vertices. The big question is: *What* vertices? These vertices cannot belong to the triangle with which we begin the construction, because by doing so we fail to interpret the duality of “extended sides”. How can the extended sides be dualized?

The surprising answer is that we take the circumscribed circle, not through the three vertices of the original triangle, but through one of its vertices and *the opposite points of the two other vertices*!

We have one more problem left: In Theorem 1 we prove that the sum of sides is constant. The dual of the length of a side between two vertices is the measure of the angle between two sides. Taking all this into account, we arrive at the following dual theorem on the sphere:

So the dual definition can be impoved: Given a triangle, and a circle which goes through a vertex of the triangle and the opposite points of the two other vertices. With the two vertices and their opposite points fixed, draw another third vertex on the arc of the circle, and get a new triangle. The sum of measures of angles in the two triangles is the same.

We can say even more: Given a spherical triangle, fix two vertices, and draw a circle through the third vertex and the opposite points of the two fixed vertices. (This circle corresponds to the inscribed circle in the supplementary triangle in Theorem 1.) Choose another third vertex on the same arc of the circle where the original third vertex was drawn. (This vertex corresponds to the new third side in Theorem 1.) Get a new triangle, prove that the sum of angles in the new triangle is the same as in the original triangle. (The sum of angles corresponds to the sum of sides in Theorem 1.)

We can make a step further, knowing that the area of a spherical triangle depends on the sum of its interior angles: Prove that the area of the new triangle is the same as that of the original triangle. The circle constructed in this way is called the Lexell circle named after Anders Johan Lexell, a Finnish scientist, close friend and helper of Leonhard Euler.

The theorem can be proved directly in absolute geometry, but if we accept the principle of duality, the proof of Theorem 1 proves the dual theorem as well.

Figure 3 shows a special case when the area of the triangles at issue is exactly one-fourth of the whole sphere. The two fixed vertices are the endpoints of the diameter of the circle. the third vertex of the inscribed triangle moves along the arc of the circle. The area of the supplementary triangle is constant, 180°in this case.

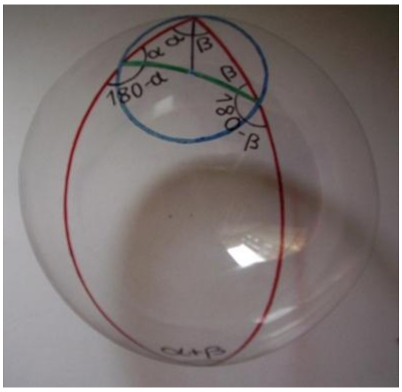

Figure 3

Conclusions. For many students, a geometric concept only means recalling the visual image of a geometric shape. The definition is only a roleplay to please the teacher at school, a text to be memorized without any relevance to the geometric figure. The true meaning of a definition becomes clear for the student if the same concept has to be interpreted in a different context, from a different perspective. The principle of duality within the same geometry and the method of comparison between the geometry of a plane and a sphere are very well suited for this purpose.

Another advantage is the variance in the degree of difficulty of the tasks. Theorem 1 is rewarding for those students who are less oriented towards mathematics, while the other parts are recommended for those who are more enthusiastic to the subject. Another advantage is the different levels of difficulty of the tasks. Theorem 1 is rewarding even for those who are less math-oriented, while the other parts are recommended for those who are more passionate about the subject.

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KONKURSY MATEMATYCZNE DROGĄ DO ROZBUDZANIA ZAINTERESOWAŃ MATEMATYCZNYCH U UCZNIÓW

Wprowadzenie. Wszyscy nauczyciele i rodzice poszukują dróg efektywnego uczenia się matematyki dla ich dzieci i uczniów. Nauczycielom i rodzicom zależy na tym, aby matematyka była dla uczniów interesująca i pociągająca, a jej uczenie się było przyjemnością. Jednocześnie chcielibyśmy, aby uczniowie poznawali matematykę nie tylko z podręczników, ale aby umieli odnaleźć ją wszędzie wokół siebie.