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## ITERATION METHOD OF EXPLICIT TYPE FOR SOLVING INCORRECT EQUATIONS OF FIRST KIND WITH APPROXIMATELY SPECIFIED OPERATOR

The article deals with the study of the explicit method of solving incorrect equations of the first kind with nonnegative self-adjoint limited operator in Hilbert space. It aims at proving the method convergence in case of a priori and a posteriori choice of the number of iterations in the basic norm of Hilbert space on the assumption of existing errors not only in the equation right-hand member but in the operator as well. Error estimation, a priori stop moment and estimate for the a posteriori moment stop are obtained.

### 1. Problem statement

Let  $H$  and  $F$  be Hilbert spaces and  $A \in L(H, F)$ , i.e.  $A$  is a linear continuous operator functioning from  $H$  to  $F$ . It is assumed that zero belongs to operator spectrum  $A$ , but it is not its characteristic constant. The following equation is solved

$$Ax = y. \quad (1)$$

The problem of searching for element  $x \in H$  by element  $y \in F$  is incorrect, for arbitrary small disturbances in the right-hand member  $y$  may result in arbitrary disturbances in solution.

Let us suppose that the accurate development  $x^* \in H$  of equation (1) exists and is the unique one. We shall search for it with the help of iteration process

$$\begin{aligned} x_{n+1} &= x_n - \alpha_{n+1}(Ax_n - y), \quad x_0 = 0, \\ \alpha_{2n+1} &= \alpha, \quad n = 0, 1, 2, \dots, \alpha_{2n+2} = \beta, \quad n = 0, 1, 2, \dots, \end{aligned} \quad (2)$$

where  $E$  is an identity operator while  $\alpha$  and  $\beta$  are an iteration parameters.

We consider that operator  $A$  and the right-hand member of equation (1) are specified approximately, i.e. approximation  $y_\delta$ ,  $\|y - y_\delta\| \leq \delta$  is known instead of  $y$ , and operator  $A_\eta$ ,  $\|A - A_\eta\| \leq \eta$  is known instead of operator  $A$ . Suppose  $0 \in Sp(A_\eta)$ ,  $Sp(A_\eta) \subseteq [0, 1]$ . Then method (2) will look

$$\begin{aligned} x_{n+1, \delta} &= x_{n, \delta} - \alpha_{n+1}(A_\eta x_{n, \delta} - y_\delta), \quad x_{0, \delta} = 0, \\ \alpha_{2n+1} &= \alpha, \quad n = 0, 1, 2, \dots, \alpha_{2n+2} = \beta, \quad n = 0, 1, 2, \dots. \end{aligned} \quad (3)$$

The case of approximate right-member of equation  $y_\delta$  and faithful operator  $A$  for the method under consideration (3) has been studied in [1–2]. It deals with a priori and a posteriori choice of a regularization parameter and the case of non-unique solution of problem (1).

Let us prove the method convergence (3) in case of a priori and a posteriori choice of a regularization parameter in solving the equation  $A_\eta x = y_\delta$  with the approximate operator  $A_\eta$  and the approximate right-hand member  $y_\delta$  and obtain estimated errors.



## 2. The a priori choice of the number of iterations

Let  $H$  equal  $F$ ,  $A = A^* \geq 0$ ,  $A_\eta = A_\eta^* \geq 0$ ,  $Sp(A_\eta) \subseteq [0, 1]$ ,  $0 < \eta \leq \eta_0$ . The iteration method (3) will be presented in the following way  $x_n = g_n(A_\eta)y_\delta$ , where  $g_n(\lambda) = \lambda^{-1} [1 - (1 - \alpha\lambda)^{n/2} (1 - \beta\lambda)^{n/2}] \geq 0$ , ( $n$  - an even). There have been obtained in [1–2] the conditions for functions  $g_n(\lambda)$  with  $0 < \alpha < 2$ ,  $(\alpha + \beta)^2 < 8\alpha\beta$ ,  $\alpha + \beta < \frac{3}{2}\alpha\beta$ ,

$$\frac{1}{16} + \alpha\beta \leq \alpha + \beta :$$

$$\sup_{0 \leq \lambda \leq 1} |g_n(\lambda)| \leq \gamma n, \quad \gamma = \frac{\alpha + \beta}{2}, \quad n > 0, \quad (4)$$

$$\sup_{0 \leq \lambda \leq 1} \lambda^s |1 - \lambda g_n(\lambda)| \leq \gamma_s n^{-s}, \quad \gamma_s = \left( \frac{s}{\alpha + \beta} \right)^s, \quad (n > 0), \quad 0 < s < \infty \quad (5)$$

(here  $s$  is the degree of source representability of exact solution  $x^* = A^s z$ ,  $s > 0$ ,  $\|z\| \leq \rho$ ),

$$\sup_{0 \leq \lambda \leq 1} |1 - \lambda g_n(\lambda)| \leq \gamma_0, \quad \gamma_0 = 1, \quad n > 0. \quad (6)$$

The following is valid:

**Lemma 1.** Let  $A = A^* \geq 0$ ,  $A_\eta = A_\eta^* \geq 0$ ,  $\|A_\eta - A\| \leq \eta$ ,  $Sp(A_\eta) \subseteq [0, 1]$ , ( $0 < \eta \leq \eta_0$ ),  $0 < \alpha < 2$ ,  $(\alpha + \beta)^2 < 8\alpha\beta$ ,  $\frac{1}{16} + \alpha\beta \leq \alpha + \beta$  and condition (6) be satisfied. Then  $\|G_{n\eta}v\| \rightarrow 0$  at  $n \rightarrow \infty$ ,  $\eta \rightarrow 0$   $\forall v \in N(A)^\perp = \overline{R(A)}$ , where  $N(A) = \{x \in H | Ax = 0\}$  and  $G_{n\eta} = E - A_\eta g_n(A_\eta)$ .

Proof. We have  $\|G_{n\eta}v\| = \|(E - A_\eta g_n(A_\eta))v\| = \left\| \int_0^1 (1 - \lambda g_n(\lambda)) dE_\lambda v \right\| = \left\| \int_0^1 (1 - \alpha\lambda)^{n/2} (1 - \beta\lambda)^{n/2} dE_\lambda v \right\| \leq \left\| \int_0^\varepsilon (1 - \alpha\lambda)^{n/2} (1 - \beta\lambda)^{n/2} dE_\lambda v \right\| + \left\| \int_\varepsilon^1 (1 - \alpha\lambda)^{n/2} (1 - \beta\lambda)^{n/2} dE_\lambda v \right\| \leq q^{n/2}(\varepsilon) \left\| \int_\varepsilon^1 dE_\lambda v \right\| \rightarrow 0, \quad n \rightarrow \infty,$

as for  $\lambda \in [\varepsilon, 1]$   $|(1 - \alpha\lambda)(1 - \beta\lambda)| \leq q(\varepsilon) < 1$ .

$\left\| \int_0^\varepsilon (1 - \alpha\lambda)^{n/2} (1 - \beta\lambda)^{n/2} dE_\lambda v \right\| \leq \left\| \int_0^\varepsilon dE_\lambda v \right\| = \|E_\varepsilon v\| \rightarrow 0, \quad \varepsilon \rightarrow 0$  owing to integrated spectrum properties [3–4]. Consequently,  $\|G_{n\eta}v\| \rightarrow 0$  at  $n \rightarrow \infty$ ,  $\eta \rightarrow 0$ . Lemma 1 is proved.

The convergence condition for method (3) is given by

**Theorem 1.** Let  $A = A^* \geq 0$ ,  $A_\eta = A_\eta^* \geq 0$ ,  $\|A_\eta - A\| \leq \eta$ ,  $Sp(A_\eta) \subseteq [0, 1]$ , ( $0 < \eta \leq \eta_0$ ),  $y \in R(A)$ ,  $\|y - y_\delta\| \leq \delta$  and conditions  $0 < \alpha < 2$ ,  $(\alpha + \beta)^2 < 8\alpha\beta$ ,



$\frac{1}{16} + \alpha\beta \leq \alpha + \beta$ , (4) be satisfied. Let us choose parameter  $n = n(\delta, \eta)$  in approximation (3)

so that  $(\delta + \eta)n(\delta, \eta) \rightarrow 0$  at  $n(\delta, \eta) \rightarrow \infty$ ,  $\delta \rightarrow 0$ ,  $\eta \rightarrow 0$ . Then  $x_{n(\delta, \eta)} \rightarrow x^*$  at  $\delta \rightarrow 0, \eta \rightarrow 0$ .

Proof. We have  $x_n = g_n(A_\eta)y_\delta$ . Then

$$\begin{aligned} x_n - x^* &= g_n(A_\eta)y_\delta - x^* = -G_{n\eta}x^* + G_{n\eta}x^* + g_n(A_\eta)y_\delta - x^* = \\ &= -G_{n\eta}x^* + (E - A_\eta g_n(A_\eta))x^* + g_n(A_\eta)y_\delta - x^* = -G_{n\eta}x^* + g_n(A_\eta)(y_\delta - A_\eta x^*). \end{aligned}$$

Condition (4) being as follows  $\|g_n(A_\eta)\| \leq \sup_{0 \leq \lambda \leq 1} |g_n(\lambda)| \leq \gamma n$ ,  $\gamma = \frac{\alpha + \beta}{2}$ , then

$$\|y_\delta - A_\eta x^*\| \leq \|y_\delta - y\| + \|y - A_\eta x^*\| = \|y_\delta - y\| + \|Ax^* - A_\eta x^*\| \leq \delta + \|A - A_\eta\| \|x^*\| \leq \delta + \eta \|x^*\|.$$

$$\text{Consequently, } \|x_{n(\delta, \eta)} - x^*\| \leq \|G_{n\eta}x^*\| + \|g_n(A_\eta)(y_\delta - A_\eta x^*)\| \leq \|G_{n\eta}x^*\| + \gamma n (\delta + \eta \|x^*\|).$$

As appears from Lemma 1,  $\|G_{n\eta}x^*\| \rightarrow 0$  at  $n \rightarrow \infty$ ,  $\eta \rightarrow 0$ , and according to the condition of Theorem 1,  $n(\delta + \eta) \rightarrow 0$  at  $\delta \rightarrow 0$ ,  $\eta \rightarrow 0$ . Thus,  $\|x_{n(\delta, \eta)} - x^*\| \rightarrow 0$ ,  $\delta \rightarrow 0$ ,  $\eta \rightarrow 0$ . Theorem 1 is proved.

**Theorem 2.** Let  $A = A^* \geq 0$ ,  $A_\eta = A_\eta^* \geq 0$ ,  $\|A_\eta - A\| \leq \eta$ ,  $Sp(A_\eta) \subseteq [0, 1]$ ,  $(0 < \eta \leq \eta_0)$ ,  $y \in R(A)$ ,  $\|y_\delta - y\| \leq \delta$  and conditions  $0 < \alpha < 2$ ,  $(\alpha + \beta)^2 < 8\alpha\beta$ ,  $\alpha + \beta < \frac{3}{2}\alpha\beta$ ,  $\frac{1}{16} + \alpha\beta \leq \alpha + \beta$ , (4), (5), (6) be satisfied. If the exact solution is source representable, i.e.  $x^* = A^s z$ ,  $s > 0$ ,  $\|z\| \leq \rho$ , then error estimation is equitable

$$\|x_{n(\delta, \eta)} - x^*\| \leq \gamma_0 c_s \eta^{\min(1,s)} \rho + \gamma_s n^{-s} \rho + \gamma n (\delta + \eta \|x^*\|), \quad 0 < s < \infty.$$

Proof. Using the source representability of the exact solution we have

$$\|G_{n\eta}x^*\| = \|G_{n\eta}A^s z\| \leq \|G_{n\eta}(A^s - A_\eta^s)z\| + \|G_{n\eta}A_\eta^s z\| \leq \gamma_0 c_s \eta^{\min(1,s)} \rho + \gamma_s n^{-s} \rho,$$

as according to Lemma 1.1 [5, p. 91]  $\|A_\eta^s - A^s\| \leq c_s \eta^{\min(1,s)}$ ,  $c_s = \text{const}$ , ( $c_s \leq 2$  for  $0 < s \leq 1$ ).

$$\text{Then } \|x_{n(\delta, \eta)} - x^*\| \leq \gamma_0 c_s \eta^{\min(1,s)} \rho + \gamma_s n^{-s} \rho + \gamma n (\delta + \eta \|x^*\|), \quad 0 < s < \infty. \quad (7)$$

Theorem 2 is proved.



If the right side of estimation (7) is minimized by  $n$ , we get the meaning of a priori

$$\text{stop moment: } n_{\text{opt}} = \left[ \frac{s\gamma_s \rho}{\gamma(\delta + \|x^*\|\eta)} \right]^{1/(s+1)} = d_s \rho^{1/(s+1)} [\delta + \eta \|x^*\|]^{-1/(s+1)}, \quad \text{where}$$

$$d_s = \left( \frac{s\gamma_s}{\gamma} \right)^{1/(s+1)}. \text{ Consequently,}$$

$$n_{\text{opt}} = s \left( \frac{\alpha + \beta}{2} \right)^{-1} 2^{-s/(s+1)} \rho^{1/(s+1)} (\delta + \eta \|x^*\|)^{-1/(s+1)}.$$

Let us substitute  $n_{\text{opt}}$  in estimation (7) to get

$$\begin{aligned} \|x_{n(\delta,\eta)} - x^*\|_{\text{opt}} &\leq \gamma_0 c_s \eta^{\min(1,s)} \rho + \gamma_s \rho (d_s \rho^{1/(s+1)})^{-s} (\delta + \eta \|x^*\|)^{s/(s+1)} + \\ &+ \gamma (\delta + \eta \|x^*\|) d_s \rho^{1/(s+1)} (\delta + \eta \|x^*\|)^{-1/(s+1)} = \\ &= \gamma_0 c_s \eta^{\min(1,s)} \rho + (\delta + \eta \|x^*\|)^{s/(s+1)} (d_s^{-s} \gamma_s \rho^{1/(s+1)} + \gamma d_s \rho^{1/(s+1)}) = \\ &= \gamma_0 c_s \eta^{\min(1,s)} \rho + \rho^{1/(s+1)} c'_s (\delta + \eta \|x^*\|)^{s/(s+1)}, \end{aligned}$$

where  $c'_s = d_s^{-s} \gamma_s + \gamma d_s = (s^{1/(s+1)} + s^{-s/(s+1)}) \gamma^{s/(s+1)} \gamma_s^{1/(s+1)} = (1+s) 2^{-s/(s+1)}$ . Hence

$$\|x_{n(\delta,\eta)} - x^*\|_{\text{opt}} \leq c_s \eta^{\min(1,s)} \rho + (1+s) 2^{-s/(s+1)} \rho^{1/(s+1)} (\delta + \eta \|x^*\|)^{s/(s+1)}.$$

**Note 1.** Optimal error estimation does not depend on  $\alpha$  and  $\beta$ , whereas  $n_{\text{opt}}$  depends on  $\alpha$  and  $\beta$ . Thus, for the contraction of computing it is necessary to take  $\alpha$  and  $\beta$  probably big of conditions  $0 < \alpha < 2$ ,  $(\alpha + \beta)^2 < 8\alpha\beta$ ,  $\alpha + \beta < \frac{3}{2}\alpha\beta$ ,  $\frac{1}{16} + \alpha\beta \leq \alpha + \beta$ , (4), (5), (6) and it, that  $n_{\text{opt}} \in \mathbb{Z}$ .

### 3. The a posteriori choice of a number of iterations.

The equation is considered  $Ax = y$  from section 1. Iteration method (3) is used for its solution.

Let's prove the convergence of a method (3) in case of a posteriori method of choice of parameter of regularization for the solution of equation  $A_\eta x = y_\delta$ , where the operator  $A_\eta$  and the right-hand member of the equation are approximately:  $\|A_\eta - A\| \leq \eta$ ,  $\|y - y_\delta\| \leq \delta$ . Such questions were studied in [5], but for other methods. We consider, that zero is not the eigenvalue of the operator  $A_\eta$ , but belongs to its spectrum. Let's consider that the equation  $A_\eta x = y_\delta$  has the only solution.

Let's set stop level  $\varepsilon > 0$  and define the moment  $m$  stop of iteration method (3) the condition



$$\left. \begin{array}{l} \|A_\eta x_{n(\delta,\eta)} - y_\delta\| > \varepsilon, (n < m), \\ \|A_\eta x_{m(\delta,\eta)} - y_\delta\| \leq \varepsilon, \end{array} \right\} \varepsilon = b(\delta + \|x^*\|\eta), b > 1. \quad (8)$$

Let's suggest that in the initial approximation  $x_{0(\delta,\eta)}$  discrepancy is big enough, more than the level of stop  $\varepsilon$ , so  $\|A_\eta x_{0(\delta,\eta)} - y_\delta\| > \varepsilon$ . Let's show the possibility of application of the rule (8) to the method (3).

Let  $H = F$ ,  $A = A^* \geq 0$ ,  $A_\eta = A_\eta^* \geq 0$ ,  $Sp(A_\eta) \subseteq [0, 1]$ ,  $0 < \eta \leq \eta_0$ . Valid

**Lemma 3.** Let  $A = A^* \geq 0$ ,  $A_\eta = A_\eta^* \geq 0$ ,  $\|A_\eta - A\| \leq \eta$ ,  $\|A_\eta\| \leq 1$  ( $n$  – an even) and the condition is satisfied (5). For  $G_{n\eta} = E - A_\eta g_n(A_\eta)$  the relation to is fair for  $\forall v \in \overline{R(A)}$ :

$$n\|A_\eta G_{n\eta} v\| \rightarrow 0 \text{ if } n \rightarrow \infty, \eta \rightarrow 0. \quad (9)$$

Proof. We will use the theorem Banach – Steingaus [4]. Here  $\|B_n\| = n\|A_\eta G_{n\eta}\|$  and on a condition (5) norms  $\|B_n\|$  are limited in total

$$n\|A_\eta G_{n\eta}\| = n\|A_\eta(E - A_\eta g_n(A_\eta))\| =$$

$$= n \sup_{0 \leq \lambda \leq 1} |\lambda| |1 - \lambda g_n(\lambda)| \leq n \gamma_1 n^{-1} = \gamma_1, (n > 0, \eta > 0).$$

For form elements  $v = A\omega$ , making in  $\overline{R(A)}$  a dense subset, by virtue of (5) we have

$$\begin{aligned} n\|A_\eta G_{n\eta} v\| &= n\|A_\eta G_{n\eta} A\omega\| \leq n\|A_\eta G_{n\eta}(A - A_\eta)\omega\| + \\ &+ n\|A_\eta G_{n\eta} A_\eta \omega\| \leq \left( \gamma_1 \eta + n \sup_{0 \leq \lambda \leq 1} \lambda^2 |1 - \lambda g_n(\lambda)| \right) \|\omega\| \leq \\ &\leq (\gamma_1 \eta + n \gamma_2 n^{-2}) \|\omega\| = (\gamma_1 \eta + \gamma_2 n^{-1}) \|\omega\| \rightarrow 0, n \rightarrow \infty, \eta \rightarrow 0. \end{aligned}$$

According to Banach – Steingaus theorem  $n\|A_\eta G_{n\eta} v\| \rightarrow 0$  at  $n \rightarrow \infty, \eta \rightarrow 0$ . Lemma 3 is proved.

**Lemma 4.** Let  $A = A^* \geq 0$ ,  $A_\eta = A_\eta^* \geq 0$ ,  $\|A_\eta - A\| \leq \eta$ ,  $\|A_\eta\| \leq 1$ , and the conditions

are satisfied (4), (6). If for some  $v_0 \in \overline{R(A)}$ ,  $n_p \leq \bar{n} = \text{const}$  and  $\eta_p \rightarrow 0$  we have  $A_{\eta_p} G_{n_p \eta_p} v_0 \rightarrow 0$  at  $p \rightarrow \infty$ , so  $G_{n_p \eta_p} v_0 \rightarrow 0$ .

Proof. Owing to an inequality (6) the sequence  $v_p = G_{n_p \eta_p} v_0$  is limited, so  $\|v_p\| = \|G_{n_p \eta_p} v_0\| \leq \gamma_0 \|v_0\|$ ,  $p \in N = \{1, 2, \dots\}$ . Thus in Hilbert space out of it sequence we can remove poorly agreed subsequence  $v_p \rightarrow v$ ,  $(p \in N' \subseteq N)$ . Then  $A_{\eta_p} v_p \rightarrow A_{\eta_p} v$ ,



$(p \in N')$ . Under the condition  $\omega_p = A_{\eta_p} v_p \rightarrow 0$ , it means,  $A_{\eta_p} v = 0$ . But as zero is not the eigenvalue of the operator  $A_{\eta_p}$ , so  $v = 0$ . And now

$$\begin{aligned} \|v_p\|^2 &= \left( v_p, G_{n_p \eta_p} v_0 \right) = \left( v_p, (E - A_{\eta_p} g_{n_p}(A_{\eta_p})) v_0 \right) = \left( v_p, v_0 \right) - \\ &- \left( A_{\eta_p} v_p, g_{n_p}(A_{\eta_p}) v_0 \right) = \left( v_p, v_0 \right) - \left( \omega_p, g_{n_p}(A_{\eta_p}) v_0 \right) \rightarrow (v, v_0) = (0, v_0) = 0, (p \in N'), \end{aligned}$$

as  $\omega_p \rightarrow 0$ , on a condition (4)  $\|g_{n_p}(A_{\eta_p})\| \leq \gamma n_p \leq \gamma \bar{n}$ . So, any weakly convergent subsequence of a limited sequence  $v_p$  aspires to zero on norm. Thus it follows, that all the sequence  $v_p \rightarrow 0, p \rightarrow \infty$  is on norm. Lemma 4 is proved.

We'll use the proved lemmms to prove the following theorems.

**Theorem 3.** Let  $H = F$ ,  $A = A^* \geq 0$ ,  $A_\eta = A_\eta^* \geq 0$ ,  $\|A_\eta - A\| \leq \eta$ ,  $\|A_\eta\| \leq 1$ ,  $(0 < \eta \leq \eta_0)$ ,  $y \in R(A)$ ,  $\|y - y_\delta\| \leq \delta$  and the conditions are satisfied  $0 < \alpha < 2$ ,  $(\alpha + \beta)^2 < 8\alpha\beta$ ,  $\frac{1}{16} + \alpha\beta \leq \alpha + \beta$ , (4), (5), (6). Let the parameter  $m(\delta, \eta)$  ( $m$ -an even) is chosen by the rule (8). So  $(\delta + \eta)m(\delta, \eta) \rightarrow 0$ ,  $x_{m(\delta, \eta)} \rightarrow x^*$  under  $\delta \rightarrow 0$ ,  $\eta \rightarrow 0$ .

Proof. We have  $x_{n(\delta, \eta)} = g_n(A_\eta)y_\delta$ , then

$$\begin{aligned} x_{n(\delta, \eta)} - x^* &= -x^* + g_n(A_\eta)y_\delta = -G_{n\eta}x^* + G_{n\eta}x^* - \\ &- x^* + g_n(A_\eta)y_\delta = -G_{n\eta}x^* + (E - A_\eta g_n(A_\eta))x^* - \\ &- x^* + g_n(A_\eta)y_\delta = -G_{n\eta}x^* + x^* - A_\eta g_n(A_\eta)x^* - \\ &- x^* + g_n(A_\eta)y_\delta = -G_{n\eta}x^* + g_n(A_\eta)(y_\delta - A_\eta x^*). \end{aligned}$$

Consequently,

$$x_{n(\delta, \eta)} - x^* = -G_{n\eta}x^* + g_n(A_\eta)(y_\delta - A_\eta x^*). \quad (10)$$

Hence

$$\begin{aligned} A_\eta x_{n(\delta, \eta)} - A_\eta x^* &= -A_\eta G_{n\eta}x^* + A_\eta g_n(A_\eta)y_\delta - A_\eta g_n(A_\eta)A_\eta x^*; \\ A_\eta x_{n(\delta, \eta)} &= A_\eta x^* - A_\eta G_{n\eta}x^* + A_\eta g_n(A_\eta)y_\delta - A_\eta g_n(A_\eta)A_\eta x^*; \\ A_\eta x_{n(\delta, \eta)} - y_\delta &= -A_\eta G_{n\eta}x^* - y_\delta + (E - A_\eta g_n(A_\eta))A_\eta x^* + \\ &+ A_\eta g_n(A_\eta)y_\delta = -A_\eta G_{n\eta}x^* + G_{n\eta}A_\eta x^* - (E - A_\eta g_n(A_\eta))y_\delta = \\ &= -A_\eta G_{n\eta}x^* + G_{n\eta}A_\eta x^* - G_{n\eta}y_\delta = -A_\eta G_{n\eta}x^* - G_{n\eta}(y_\delta - A_\eta x^*). \end{aligned}$$

Thus,

$$A_\eta x_{n(\delta, \eta)} - y_\delta = -A_\eta G_{n\eta}x^* - G_{n\eta}(y_\delta - A_\eta x^*). \quad (11)$$

According to lemma 1 it follows that

$$\|G_{n\eta}x^*\| \rightarrow 0, n \rightarrow \infty, \eta \rightarrow 0. \quad (12)$$

Let's show that



$$\|g_n(A_\eta)(y_\delta - A_\eta x^*)\| \leq \gamma n (\delta + \|x^*\|\eta). \quad (13)$$

On a condition (4)  $\|g_n(A_\eta)\| \leq \sup_{0 \leq \lambda \leq 1} |g_n(\lambda)| \leq \gamma n$ , and  $\|y_\delta - A_\eta x^*\| \leq \|y_\delta - y\| + \|y - A_\eta x^*\| = \|y_\delta - y\| + \|Ax^* - A_\eta x^*\| \leq \delta + \|(A - A_\eta)x^*\| \leq \delta + \|x^*\|\eta$ , thus  $\|g_n(A_\eta)(y_\delta - A_\eta x^*)\| \leq \gamma n (\delta + \|x^*\|\eta)$ .

By virtue of lemma 3

$$\sigma_{m\eta} = n \|A_\eta G_{m\eta} x^*\| \rightarrow 0, n \rightarrow \infty, \eta \rightarrow 0. \quad (14)$$

We will apply the rule of stop (8), then  $\|A_\eta x_{m(\delta, \eta)} - y_\delta\| \leq b(\delta + \|x^*\|\eta)$ ,  $b > 1$  and from (6) and (11) we get

$$\|A_\eta G_{m\eta} x^*\| \leq (b+1)(\delta + \|x^*\|\eta). \quad (15)$$

$$\begin{aligned} \text{Reall, from (11)} \quad & \|A_\eta G_{m\eta} x^*\| \leq \|A_\eta x_{m(\delta, \eta)} - y_\delta\| + \|G_{m\eta}(y_\delta - A_\eta x^*)\| \leq \\ & \leq b(\delta + \|x^*\|\eta) + (\delta + \|x^*\|\eta) = (b+1)(\delta + \|x^*\|\eta). \end{aligned}$$

For  $\forall n < m$   $\|A_\eta x_{n(\delta, \eta)} - y_\delta\| > \varepsilon$ , consequently

$$\|A_\eta G_{n\eta} x^*\| \geq \|A_\eta x_{n(\delta, \eta)} - y_\delta\| - \|G_{n\eta}(y_\delta - A_\eta x^*)\| \geq (b-1)(\delta + \|x^*\|\eta).$$

Thus, for  $\forall n < m$

$$\|A_\eta G_{n\eta} x^*\| \geq (b-1)(\delta + \|x^*\|\eta). \quad (16)$$

From (16) and (14) we have  $\frac{\sigma_{m-2,\eta}}{m-2} = \|A_\eta G_{m-2,\eta} x^*\| \geq (b-1)(\delta + \|x^*\|\eta)$  under  $n = m-2$  or  $(m-2)(\delta + \|x^*\|\eta) \leq \frac{\sigma_{m-2,\eta}}{b-2} \rightarrow 0, \delta \rightarrow 0, \eta \rightarrow 0$  (because from (14)  $\sigma_{m\eta} \rightarrow 0, m \rightarrow \infty, \eta \rightarrow 0$ ). If it  $m(\delta, \eta) \rightarrow \infty$  under  $\delta \rightarrow 0, \eta \rightarrow 0$ , so, using (10), (12)

$$\begin{aligned} \text{and (13), we get} \quad & \|x_{m(\delta, \eta)} - x^*\| \leq \|G_{m\eta} x^*\| + \|g_m(A_\eta)(y_\delta - A_\eta x^*)\| \leq \\ & \leq \|G_{m\eta} x^*\| + \gamma m(\delta, \eta)(\delta + \|x^*\|\eta) \rightarrow 0, m \rightarrow \infty, \delta \rightarrow 0, \eta \rightarrow 0, \end{aligned}$$

so that  $x_{m(\delta, \eta)} \rightarrow x^*$ .

If for any  $\delta_n$  and  $\eta_n$  the consequence  $m(\delta_n, \eta_n)$  will be limited, even in this case  $x_{m(\delta_n, \eta_n)} \rightarrow x^*, \delta_n \rightarrow 0, \eta_n \rightarrow 0$ . It is true, out of (15) is made



$\|A_{\eta_n} G_{m\eta_n} x^*\| \leq (b+1) \left( \delta_n + \|x^*\| \eta_n \right) \rightarrow 0, \delta_n \rightarrow 0, \eta_n \rightarrow 0.$  Thus, we have  $A_{\eta_n} G_{m\eta_n} x^* \rightarrow 0, \delta_n \rightarrow 0, \eta_n \rightarrow 0$  and under lemma 4 we get that under  $\delta_n \rightarrow 0, \eta_n \rightarrow 0$  it is made  $G_{m\eta_n} x^* \rightarrow 0.$  Thus

$$\|x_{m(\delta_n, \eta_n)} - x^*\| \leq \|G_{m\eta_n} x^*\| + \gamma m(\delta_n, \eta_n) \left( \delta_n + \|x^*\| \eta_n \right) \rightarrow 0, \delta_n \rightarrow 0, \eta_n \rightarrow 0.$$

Theorem 3 is proved.

**Theorem 4.** Let the following theorem 3 conditions are satisfied. If  $x^* = A^s z, s > 0, \|z\| \leq \rho,$  that are true the assessment

$$m \leq 2 + \frac{s+1}{\alpha+\beta} \frac{\rho^{1/(s+1)}}{\left[ (b-1) \left( \delta + \|x^*\| \eta \right) - c_s \gamma_1 \eta \rho \right]^{1/(s+1)}},$$

$$\begin{aligned} \|x_{m(\delta, \eta)} - x^*\| &\leq c_s \eta^{\min(1, s)} \rho + \left[ c_s \gamma_1 \eta \rho + (b+1) \left( \delta + \|x^*\| \eta \right) \right]^{s/(s+1)} \rho^{1/(s+1)} + \\ &+ \frac{\alpha+\beta}{2} \left\{ 2 + \frac{s+1}{\alpha+\beta} \frac{\rho^{1/(s+1)}}{\left[ (b-1) \left( \delta + \|x^*\| \eta \right) - c_s \gamma_1 \eta \rho \right]^{1/(s+1)}} \right\} \left( \delta + \|x^*\| \eta \right). \end{aligned} \quad (17)$$

Proof. Let's value again  $\|A_\eta G_{m-2, \eta} x^*\|.$  By virtue of (5) and lemma 1.1 [5, p. 91]

$$\begin{aligned} \|A_\eta G_{m-2, \eta} x^*\| &= \|A_\eta G_{m-2, \eta} A^s z\| \leq \|A_\eta G_{m-2, \eta} (A^s - A_\eta^s) z\| + \\ &+ \|A_\eta^{s+1} G_{m-2, \eta} z\| \leq (\beta_{m-2, s} \eta + \gamma_{s+1} (m-2)^{-(s+1)}) \rho, \end{aligned}$$

where  $\beta_{m-2, s} = c_s \sup_{0 \leq \lambda \leq 1} \lambda (1 - \lambda g_{m-2}(\lambda)) \leq [(m-2)(\alpha+\beta)]^{-1} c_s = c_s \gamma_1 (m-2)^{-1}, \beta_{m-2, s} \rightarrow 0,$   $m \rightarrow \infty.$  Here  $c_s = \text{const}$  ( $c_s \leq 2$  under  $0 < s \leq 1$ ). Comparing it with (16), we get  $(b-1) \left( \delta + \|x^*\| \eta \right) \leq (\beta_{m-2, s} \eta + \gamma_{s+1} (m-2)^{-(s+1)}) \rho.$  Hence we have

$$(m-2)^{(s+1)} \leq \frac{\gamma_{s+1} \rho}{\left[ (b-1) \left( \delta + \|x^*\| \eta \right) - \beta_{m-2, s} \eta \rho \right]}, \text{ and, consequently,}$$

$$m \leq 2 + \frac{s+1}{\alpha+\beta} \left[ \frac{\rho}{(b-1) \left( \delta + \|x^*\| \eta \right) - \beta_{m-2, s} \eta \rho} \right]^{1/(s+1)}.$$



As  $\beta_{m-2,s} = c_s \gamma_1 \frac{1}{m-2} \leq c_s \gamma_1$  (as at  $m > 2$   $\frac{1}{m-2} \leq 1$ ),

so  $(b-1)\left(\delta + \|x^*\|\eta\right) - \beta_{m-2,s}\eta\rho \geq (b-1)\left(\delta + \|x^*\|\eta\right) - c_s \gamma_1 \eta\rho$ , and, it means, we get the following assessment for  $m$ :  $m \leq 2 + \frac{s+1}{\alpha+\beta} \frac{\rho^{1/(s+1)}}{\left[(b-1)\left(\delta + \|x^*\|\eta\right) - c_s \gamma_1 \eta\rho\right]^{1/(s+1)}}$ .

We have  $\|G_{m\eta}x^*\| = \|G_{m\eta}A^s z\| \leq \|G_{m\eta}(A^s - A_\eta^s)z\| + \|G_{m\eta}A_\eta^s z\|$ . Under lemma 1.1 [5, p. 91]  $\|G_{m\eta}(A^s - A_\eta^s)z\| \leq c_s \eta^{\min(1,s)} \rho$ , that gives the contribution  $O((\delta + \eta)^{s/(s+1)})$  to an assessment  $\|x_{m(\delta,\eta)} - x^*\|$  [5, p. 111]. We will estimate norm  $\|G_{m\eta}A_\eta^s z\|$  with the help of inequality of moments, lemmas 1.1 [5, c. 91] and (15):

$$\begin{aligned} \|G_{m\eta}A_\eta^s z\| &= \|A_\eta^s G_{m\eta} z\| = \|A_\eta^{s+1} G_{m\eta} z\|^{s/(s+1)} \|G_{m\eta} z\|^{1/(s+1)} \leq \\ &\leq \|A_\eta G_{m\eta} A_\eta^s z\|^{s/(s+1)} \|z\|^{1/(s+1)} \leq \left( \|A_\eta G_{m\eta}(A_\eta^s - A^s)z\| + \right. \\ &\quad \left. + \|A_\eta G_{m\eta} A^s z\| \right)^{s/(s+1)} \rho^{1/(s+1)} \leq \left[ \beta_{ms} \eta \rho + (b+1) \left( \delta + \|x^*\| \eta \right) \right]^{s/(s+1)} \rho^{1/(s+1)}. \end{aligned}$$

Then

$$\begin{aligned} \|x_{m(\delta,\eta)} - x^*\| &\leq \|G_{m\eta}x^*\| + \|g_m(A_\eta)(y_\delta - A_\eta x^*)\| \leq c_s \eta^{\min(1,s)} \rho + \left[ \beta_{ms} \eta \rho + (b+1) \times \right. \\ &\quad \times \left. \left( \delta + \|x^*\| \eta \right) \right]^{s/(s+1)} \rho^{1/(s+1)} + \gamma m \left( \delta + \|x^*\| \eta \right) \leq c_s \eta^{\min(1,s)} \rho + \\ &\quad + \left[ c_s \gamma_1 \eta \rho + (b+1) \left( \delta + \|x^*\| \eta \right) \right]^{s/(s+1)} \rho^{1/(s+1)} + \\ &\quad + \frac{\alpha+\beta}{2} \left\{ 2 + \frac{s+1}{\alpha+\beta} \frac{\rho^{1/(s+1)}}{\left[ (b-1) \left( \delta + \|x^*\| \eta \right) - c_s \gamma_1 \eta \rho \right]^{1/(s+1)}} \right\} \left( \delta + \|x^*\| \eta \right). \end{aligned}$$

Theorem 4 is proved.

**Note 2.** The order of evaluation (17) is  $O((\delta + \eta)^{s/(s+1)})$ , and, as it comes from [5], is optimum in a class of problems with source representable decisions.

**Note 3.** Though the formulation of theorem 4 is given with the indication of degree of source representable  $s$  and source representable element  $z$ , in practice their value are not required, because they are not held in the stopping rule for the discrepancy (8).



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**Матысик О.В. Итерационный метод явного типа для решения некорректных уравнений первого рода с приближенно заданным оператором**

*Статья посвящена изучению явного метода решения некорректных уравнений первого рода с неотрицательным самосопряженным ограниченным оператором в гильбертовом пространстве. Доказана сходимость метода в случае априорного и апостериорного выбора числа итераций в исходной норме гильбертова пространства, в предположении, что погрешности есть не только в правой части уравнения, но и в операторе. Получены оценка погрешности метода, априорный момент останова и оценка для апостериорного момента останова.*