Short communications

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ON MAXIMAL SUBGROUP OF A FINITE SOLVABLE GROUP

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Abstract. Let H be a non-normal maximal subgroup of a finite solvable group G, and let $q \in \pi(F(H/\mathrm{Core}_G H))$. It is proved that G has a Sylow q-subgroup Q such that $N_G(Q) \subseteq H$.

1 Introduction

All groups considered in this paper are finite. All notation and definitions correspond to those in [1, 2].

In 1986, V. A. Vedernikov obtained the following result:

Theorem A. [3, Corollary 2.1] *If H is a non-normal maximal subgroup of a solvable group G then $N_G(Q) \subseteq H$ for some Sylow subgroup Q of G.*

Here $N_G(Q)$ is the normalizer of Q in G.

In this paper we consider the following problem:

What is a Sylow subgroup such that its normalizer is contained in a non-normal maximal subgroup of a solvable group?

Answering the question we prove the following theorem:

Theorem 1.1. *Let H be a non-normal maximal subgroup of a solvable group G, and let $q \in \pi(F(H/\mathrm{Core}_G H))$. Then G has a Sylow q-subgroup Q such that $N_G(Q) \subseteq H$.*

Here $F(X)$ is the Fitting subgroup of X, $\pi(Y)$ is the set of all prime divisors of $|Y|$, $\mathrm{Core}_G H = \cap_{g \in G} H^g$ is the core of H in G, i. e., is the largest normal subgroup of G contained in H.

Corollary 1.1. *Let H be a non-normal maximal subgroup of a solvable group G, and let $q \in \pi(F(H/\mathrm{Core}_G H))$. Then H has a Sylow q-subgroup Q such that $N_G(H_1) \subseteq H$ for each subgroup H_1 of H satisfying $Q \subseteq H_1 \subseteq H$.*

Corollary 1.2. *Let H be a non-normal maximal subgroup of a solvable group G, and let $\omega \subset \pi(F(H/\mathrm{Core}_G H))$. Then G has a Hall ω-subgroup G_ω such that $N_G(G_\omega) \subseteq H$.*

For non-solvable groups, this result is false. For example, $PSL(2,17)$ has the order $2^4 \cdot 3^2 \cdot 17$, and the symmetric group S_4 is a maximal subgroup in $PSL(2,17)$, see [3]. Since $|S_4| = 2^3 \cdot 3$, it follows that S_4 does not contain a Sylow subgroup of $PSL(2,17)$. Thus it is not possible to extend theorem of V. A. Vedernikov and Theorem 1 to non-solvable groups.

The following question is contained in [3]:

Question. (V. A. Vedernikov, [3]) *Is it possible to extend Theorem A to a p-solvable group G containing a maximal subgroup M such that $|G : M| = p^a$, $a \in \mathbb{N}$?*

Answering this question we prove the following theorem:

Theorem 1.2. *Let G be a p-solvable group. Let M be a non-normal maximal subgroup of G, and let $|G : M| = p^a$, $a \in \mathbb{N}$. Then:*

1) *if $F(M/\mathrm{Core}_G M) \neq 1$ and $q \in \pi(F(M/\mathrm{Core}_G M))$ then G has a Sylow q-subgroup Q such that $N_G(Q) \subseteq M$;*

2) *if $F(M/\mathrm{Core}_G M) = 1$ then $N_G(K) \subseteq M$ for some Hall p'-subgroup K of G.*

2 Notations and preliminary results

In this section we give some definitions and basic results that will be used in our paper.

Let $\mathbb{P}$ be a set of all prime numbers, and let π be a set of primes, i. e., $\pi \subseteq \mathbb{P}$. In the paper, π' is the set of all primes not contained in π, i. e., $\pi' = \mathbb{P} \setminus \pi$, $\pi(m)$ is the set of all prime divisors of m. If $\pi(m) \subseteq \pi$ then m is called a *π-number*.

A subgroup H of G is called a *π-subgroup*, if $|H|$ is a π-number. A subgroup H of G is called a Hall *π-subgroup*, if $|H|$ is a π-number and $|G : H|$ is a π'-number. As usual, $O_\pi(X)$ is the largest normal π-subgroup of X. A group is called *π-separable* if it has a normal series whose factors are π-groups or π'-groups.

A group G is called *π-solvable* if it is π-separable and it has a solvable Hall π-subgroup.

Lemma 2.1. *[3, Theorem 1] Let G be a π-separable group, and let H be a subgroup of G. If $|G : H|$ is a π-number then $O_\pi(H) \subseteq O_\pi(G)$.*

Lemma 2.2. *Let R be a Hall π-subgroup of a π-separable group G, and let N be a normal subgroup of G. Then $N_G(R)N/N = N_{G/N}(RN/N)$.*

Proof. For $x \in N_G(R)$ we have:

$$(x^{-1}N)(RN/N)(xN) = R^x N/N = RN/N,$$

i. e., $N_G(R)N/N \leqslant N_{G/N}(RN/N)$.

Conversely, if $yN \in N_{G/N}(RN/N)$ then $R^y N = RN$. Next R and R^y are Hall subgroups of RN which are conjugate, i. e., $R^y = R^{ak} = R^k$ for some $ak \in RN$,

$a \in R$, $k \in N$. Then $yk^{-1} \in N_G(R)$, whence $y \in N_G(R)N$, i.e., $N_{G/N}(RN/N) \leqslant N_G(R)N/N$. □

Lemma 2.3. *Let G be a π-solvable group containing a nilpotent Hall π-subgroup. If H is a maximal subgroup of G and $|G : H|$ is π-number then $O_\pi(H)$ is a normal subgroup of G.*

Proof. By Lemma 2.1, $O_\pi(H) \subseteq O_\pi(G)$. If $O_\pi(H) = O_\pi(G)$ then $O_\pi(H)$ is a normal subgroup of G. Let $O_\pi(H)$ be a proper subgroup of $O_\pi(G)$, and let G_π be a Hall π-subgroup of G. Clearly, $O_\pi(G)$ is a proper subgroup of G_π. Since G_π is a nilpotent subgroup we have that $O_\pi(H)$ is a proper subgroup of $D = N_{G_\pi}(O_\pi(H))$. Since $O_\pi(H) = H \cap O_\pi(G)$ it follows that D is not contained in H, so $N_G(O_\pi(H)) \supseteq \langle H, D\rangle = G$ and $O_\pi(H)$ is a normal subgroup of G. □

3 Main results

Theorem 3.1. *Let G be a π-solvable group containing a nilpotent Hall π-subgroup. Let M be a non-normal maximal subgroup of G, and let $|G : M|$ is a π-number. Then:*

1) *if $F(M/\mathrm{Core}_G M) \neq 1$ and $q \in \pi(F(M/\mathrm{Core}_G M))$ then G has a Sylow q-subgroup Q such that $N_G(Q) \subseteq M$;*

2) *if $F(M/\mathrm{Core}_G M) = 1$ then $N_G(K) \subseteq M$ for some Hall π'-subgroup K of G.*

Proof. Case 1: $\mathrm{Core}_G M = 1$. Since M is a π-solvable group we have that M contains a Hall π'-subgroup K. Thus

$$|G : K| = |G : M||M : K|.$$

Since $|G : M|$ is a π-number we have that K is a Hall π'-subgroup of G. Hence $O_{\pi'}(G) \leqslant K \leqslant M$, so $O_{\pi'}(G) \leqslant \mathrm{Core}_G M = 1$. Since G is a π-solvable group we have that $O_\pi(G) \neq 1$. However, G is a primitive group with a maximal subgroup M such that $\mathrm{Core}_G M = 1$. Therefore, for some $p \in \pi(G)$ we have the following:

$$N = O_p(G) = F(G) = C_G(O_p(G)) \neq 1,\ G = M[O_p(G)],\ \Phi(G) = 1,$$

and $O_p(M) = 1$.

Assume that $F(M) \neq 1$, $q \in \pi(F(M))$, and let Q be a Sylow q-subgroup of M. Then $O_q(M) \neq 1$, $O_q(M) \subseteq Q$, and since $\mathrm{Core}_G M = 1$ we have $N_G(O_q(M)) = M$. Since $O_p(M) = 1$, it follows that p does not belong to $\pi(F(M))$ and $p \neq q$. A subgroup $D = N_G(Q) \cap O_p(G)$ is a normal subgroup of $N_G(Q)$, and

$$D \subseteq C_G(Q) \subseteq C_G(O_q(M)) \subseteq N_G(O_q(M)) = M.$$

Next $D = N_G(Q) \cap O_p(G) \subseteq M \cap O_p(G) = 1$. We consider the subgroup $L = N_G(Q)O_p(G)$. Since $N_G(Q) \cap O_p(G) = 1$ we have $L = [O_p(G)]N_G(Q)$. It follows from $G = [O_p(G)]M$ and Dedekind's identity that

$$L = [O_p(G)]N_G(Q) = [O_p(G)](L \cap M),\ L/O_p(G) \simeq N_G(Q) \simeq L \cap M,$$

and $L\cap M$ is a q-closed subgroup. The inclusion $Q\subseteq L\cap M$ implies that Q is a normal subgroup of $L\cap M$. It follows from $|N_G(Q)|=|L\cap M|$ that $L\cap M=N_G(Q)$ and $N_G(Q)\subseteq M$.

Assume that $F(M)=1$, and let K be a Hall π'-subgroup of M. By Lemma 2.3, $O_\pi(M)$ is a normal subgroup of G. Hence $O_\pi(M)\leqslant \mathrm{Core}_G M=1$. Since G is a π-solvable group we have $O_{\pi'}(M)\neq 1$ and $O_{\pi'}(M)\subseteq K$. Since $\mathrm{Core}_G M=1$ we have $N_G(O_{\pi'}(M))=M$. A subgroup $B=N_G(K)\cap O_p(G)$ is a normal subgroup of $N_G(K)$ and

$$B\subseteq C_G(K)\subseteq C_G(O_{\pi'}(M))\subseteq N_G(O_{\pi'}(M))=M.$$

Next $B=N_G(K)\cap O_p(G)\subseteq M\cap O_p(G)=1$. We consider the subgroup $T=N_G(K)O_p(G)$. Since $N_G(K)\cap O_p(G)=1$ we have $T=[O_p(G)]N_G(K)$. It follows from $G=[O_p(G)]M$ and Dedekind's identity that

$$T=[O_p(G)]N_G(K)=[O_p(G)](T\cap M),\ T/O_p(G)\simeq N_G(K)\simeq T\cap M,$$

and that $T\cap M$ is a π'-closed subgroup. Since the Hall π'-subgroup K is contained in $T\cap M$, K is a normal subgroup of $T\cap M$ and $T\cap M\subseteq N_G(K)$. It follows from isomorphism $N_G(K)\simeq T\cap M$ that $|N_G(K)|=|T\cap M|$, so $T\cap M=N_G(K)$ and $N_G(K)\subseteq M$.
Thus the theorem in Case 1 is proved.

Case 2: $N=\mathrm{Core}_G M\neq 1$. We consider the quotient group $\overline{G}=G/N$. Clearly, $\overline{G}$ is a π-solvable group and $|\overline{G}:\overline{M}|=|G:M|$ is a π-number, where $\overline{M}=M/N$. Since $\mathrm{Core}_{\overline{G}}\overline{M}=1$, for the group $\overline{G}$ with a non-normal maximal subgroup $\overline{M}$, we can apply Case 1.

Assume that $F(\overline{M})\neq 1$, and let $q\in\pi(F(\overline{M}))$. By Case 1, $\overline{G}$ has a Sylow q-subgroup $\overline{Q}$ such that $N_{\overline{G}}(\overline{Q})\subseteq\overline{M}$. Let $\overline{Q}=A/N$, and let Q be a Sylow q-subgroup of A. Then $QN/N=A/N=\overline{Q}$, and

$$N_{\overline{G}}(\overline{Q})=N_{G/N}(QN/N)=N_G(Q)N/N.$$

By the condition $N_G(Q)N/N\subseteq\overline{M}=M/N$ it follows that $N_G(Q)\subseteq M$.

Assume that $F(M/\mathrm{Core}_G M)=1$. By Case 1, $N_{\overline{G}}(\overline{K})\subseteq\overline{M}$ for some Hall π'-subgroup $\overline{K}$ of $\overline{G}$. Let $\overline{K}=B/N$, and let R be a Hall π'-subgroup of B. It exists because B is a π-solvable group. Therefore R is a Hall π'-subgroup of G, $RN/N=B/N=\overline{K}$, and by Lemma 2.2,

$$N_G(R)N/N\subseteq N_{G/N}(RN/N)=N_{\overline{G}}(\overline{K}).$$

By induction, $N_G(R)N/N\subseteq\overline{M}=M/N$, so $N_G(R)\subseteq M$.
Thus the theorem is proved in Case 2. □

Note that Theorem 1.2 follows from Theorem 3.1 if $\pi=\{p\}$. If G is a solvable group then under the assumptions of Theorem 3.1 we have $\pi(F(M/\mathrm{Core}_G M))\neq\varnothing$. So Theorem 1.1 follows from Theorem 3.1.

Proof of Corollary 1.1. By Theorem 1.1, G has a Sylow q-subgroup Q such that $N_G(Q) \subseteq H$. Let H_1 be any subgroup such that $Q \subseteq H_1 \subseteq H$, and let $T = N_G(H_1)$. By Frattini's argument, we have

$$T = N_T(Q)H_1 \subseteq N_G(Q)H_1 \subseteq H, \; T = N_G(H_1) \subseteq H. \qquad \square$$

In the case in which H_1 is a Hall ω-subgroup of H we obtain Corollary 1.2.

References

[1] K. Doerk, T. Hawkes, *Finite soluble groups.* Berlin; New York: de Gruyter. 1992.

[2] B. Huppert, *Endliche Gruppen I.* Berlin, Heidelberg, New York. 1967.

[3] V.A. Vedernikov, *On π-properties of a finite groups* In the book: Arithmetic and subgroup structure of finite groups. Minsk: Science and technics. 1986, 13–19 (in Russian).

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