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О КОНЕЧНЫХ π-РАЗРЕШИМЫХ ГРУППАХ БЕЗ ШИРОКИХ ПОДГРУПП

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ON FINITE π-SOLUBLE GROUPS WITH NO WIDE SUBGROUPS

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Подгруппу будем называть широкой, если ее порядок делится на каждый простой делитель порядка всей группы. Получено строение конечных π-разрешимых групп, не содержащих широких максимальных подгрупп, индекс которых есть π-число. Исследуются группы с нильпотентными широкими подгруппами.

Ключевые слова: *конечные группы*, *π-разрешимые группы*, *нильпотентные группы.*

A subgroup *H* of a finite group *G* is said to be wide if each prime divisor of the order *G* divides the order *H*. We obtain the description of finite π-soluble groups with no wide maximal subgroups with π-number indices. We also investigate groups with π-special subgroups.

Keywords: *finite groups*, *π-soluble groups*, *nilpotent groups.*

Introduction

All groups in this paper are finite. Let G be a group. We use $\pi(G)$ to denote the set of all prime devisors of $|G|$. By $|\pi(G)|$ we denote a number of different prime devisors of $|G|$. We also use $M<\cdot G$ to denote that M is a maximal subgroup of G.

A subgroup H of a group G is said to be wide if $\pi(H)=\pi(G)$. In soluble groups maximal subgroups have primary indices. Therefore in soluble groups a non-wide maximal subgroup is a Hall subgroup. And conversely, every Hall maximal subgroup of a soluble group is not wide. V.S. Monakhov [3], N.V. Maslova and D.O. Revin [4]–[5] investigated groups all whose maximal subgroups are Hall subgroups. Simple groups with wide subgroups were enumerated in [6]. Thus, the questions of V.S. Monakhov in the Kourovka notebook [7] were solved in full.

If a group G has no wide subgroups and $k=\max_{M<G}|\pi(M)|$, then G is called quasi-k-primary. A quasi-1-primary is also called quasiprimary, and a quasi-2-primary group is also called quasibiprimary group. The order of a nilpotent quasiprimary group is equal to pq, where p and q are different primes. The order of a nonnilpotent quasiprimary group is equal to p^aq, its Sylow p-subgroup is a minimal normal subgroup, and a is the minimal positive integer such that q divides p^a-1. It is followed from Shmidt theorem [8] of groups with nilpotent proper subgroups.

S.S. Levischenko [9] investigated quasibiprimary groups. He proved that a soluble quasibiprimary group G can be represented as the semidirect product $[P]M$ of an elementary abelian Sylow p-subgroup P and a quasibiprimary maximal subgroup M of G [9, Theorem 3.1]. In a nonsoluble quasibiprimary group G the Frattini subgroup $\Phi(G)$ is primary [9, Theorem 3.2], the factor group $G/\Phi(G)$ is simple, and all such groups are enumerated [9, Theorem 2.1].

Let π be a fixed set of primes. We consider the class $\mathfrak{X}(\pi)$ of all π-soluble groups G which have no wide maximal subgroups with π-number indices:

$$\mathfrak{X}(\pi)=\{G\in\pi\mathfrak{S}:|\pi(M)|<|\pi(G)|,\ \forall M<G,\pi(G:M)\subseteq\pi\}.$$

Here $\pi\mathfrak{S}$ is the class of all π-soluble groups.

In this paper we obtain the properties of the class $\mathfrak{X}(\pi)$ and describe the structure of groups from this class. Furthermore, we prove that the factor group of a π-soluble group by its hypercenter belongs to $\mathfrak{X}(\pi)$ under certain conditions.

1 Preliminaries

If $\pi(m)\subseteq\pi$, then a positive integer m is called π-number. A group G is called π-group if $\pi(G)\subseteq\pi$, and π'-group if $\pi(G)\subseteq\pi'$. A group is called π-soluble if it has a subnormal series whose factors are either soluble π-groups or π'-groups.

All unexplained notations and terminology are standard. The reader is referred to [1], [2] if necessary.

The following properties of π-soluble groups are well known [10].

***Lemma* 1.1.** *Let G be a π-soluble group. The following assertions hold.*

(1) *G is π_1-soluble for every $\pi_1 \subseteq \pi$.*

(2) *In G there exist π-Hall and π'-Hall subgroups.*

(3) *In G there exist $\pi \cup \{r\}$-Hall subgroups for every $r \in \pi'$.*

(4) *In G there exist q'-Hall subgroups for every $q \in \pi$.*

Recall that $O_\pi(G)$ and $O_{\pi'}(G)$ are the unique largest normal π-subgroup and the unique largest normal π'-subgroup of a group G, respectively.

***Lemma* 1.2.** *Let G be a π-soluble group. The following assertions hold.*

(1) *If N is a minimal normal subgroup of G, then N is either an elementary abelian p-subgroup for some $p \in \pi$ or a π'-subgroup*, [3, Lemma 1].

(2) *If M is a maximal subgroup of G, then either $|G:M| = p^m$ for some $p \in \pi$ and positive integer m or $\pi(G:M) \subseteq \pi'$*, [3, Lemma 1].

(3) *If $\pi \cap \pi(G) \neq \varnothing$, then $O_\pi(G/O_{\pi'}(G)) \neq 1$; if $\pi' \cap \pi(G) \neq \varnothing$, then $O_{\pi'}(G/O_\pi(G)) \neq 1$*, [10].

***Lemma* 1.3.** *Let G be a π-soluble group. Then for every $q \in \pi \cap \pi(G)$ in G there exists a maximal subgroup M such that $|G:M| = q^m$ for some positive integer m.*

Proof. By Lemma 1.1 (1), a group G is q-soluble, and so by Lemma 1.1 (2) in G there exists a q'-Hall subgroup H. If M is a maximal subgroup of G containing H, then $|G:M| = q^m$ for some positive integer m. Lemma is proved.

We define the core of a subgroup H of a group G by $H_G = \bigcap_{x \in G} x^{-1}Hx$. Clearly, H_G is the largest normal subgroup of G contained in H. A group is called primitive if it has a maximal subgroup with the trivial core.

***Lemma* 1.4** [1, Theorem 4.40].

(1) *The Fitting subgroup of a primitive group is either trivial or a minimal normal subgroup.*

(2) *A nontrivial nilpotent normal subgroup of a primitive group coincides with the Fitting subgroup and it is a minimal normal subgroup.*

(3) *The Frattini subgroup of a soluble primitive nontrivial group is trivial and the Fitting subgroup is a minimal normal subgroup.*

A formation is a class of groups $\mathfrak{F}$ with the following two properties:

(i) if $G \in \mathfrak{F}$ and $N \lhd G$, then $G/N \in \mathfrak{F}$;

(ii) if N_1 and N_2 are normal subgroups of G and $G/N_1, G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

It is easy to prove the following result.

***Lemma* 1.5.** *Let $\mathfrak{F}$ be a saturated formation and G a group. If $G \notin \mathfrak{F}$ but $G/N \in \mathfrak{F}$ for every $N \lhd G$, $N \neq 1$, then G is primitive.*

By G_π and $G_{\pi'}$ we denote the π-Hall and π'-Hall subgroups of G, respectively. In particular, G_p denotes a Sylow p-subgroup of G.

***Lemma* 1.6.** *Let G be a π-soluble group. If every maximal subgroup with π-number index is normal in G, then $G = G_\pi[G_{\pi'}]$ and G_π is nilpotent. Conversely, if $G = G_\pi[G_{\pi'}]$ and G_π is nilpotent, then every maximal subgroup with π-number index is normal in G.*

Proof. By Theorem VI.9.3 [2], G is π-supersoluble. By $\mathfrak{N}_\pi$ we denote the formation of all nilpotent π-group, $\mathfrak{E}_{\pi'}$ denotes the formation of all π'-group, and $\mathfrak{F} = \mathfrak{E}_{\pi'}\mathfrak{N}_\pi$ is their formation product. Then $\mathfrak{F}$ is a saturated formation and $G \in \mathfrak{F}$ if and only if $G = G_\pi[G_{\pi'}]$.

Assume that every maximal subgroup with π-number index is normal in G. If $M < G$ and $\pi(G:M) \subseteq \pi$, then M is normal in G. Therefore $|G:M| = p \in \pi$.

Now we prove $G \in \mathfrak{F}$. Suppose that it is not true. If $X \neq 1$ is normal in G, then $G/X \in \mathfrak{F}$ by induction. In view of Lemma 1.5, G is primitive and $\Phi(G) = O_{\pi'}(G) = 1$. Let N be a minimal normal subgroup of G. Then $|N| = q \in \pi$ and $G = [N]H$, where H is a maximal subgroup with the trivial core. By hypotheses, H is normal in G, a contradiction.

Suppose that $G = G_\pi[G_{\pi'}]$ and G_π is nilpotent. If M is a maximal subgroup of G with π-number index, then $G_{\pi'} \subseteq M$ and $M/G_{\pi'}$ is a maximal subgroup of $G/G_{\pi'} \simeq G_\pi$. Hence $M/G_{\pi'}$ is normal in $G/G_{\pi'}$ since G_π is nilpotent, and so M is normal in G. Lemma is proved.

2 Properties of the class $\mathfrak{X}(\pi)$

***Lemma* 2.1.** (1) *$\mathfrak{X}(\pi)$ is a saturated homomorph.*

(2) *If $\pi_1 \subseteq \pi$, then $\mathfrak{X}(\pi) \subseteq \mathfrak{X}(\pi_1)$.*

(3) *If $G \in \mathfrak{X}(\pi(G))$ and $\pi(G) \subseteq \pi$, then $G \in \mathfrak{X}(\pi)$.*

(4) *Let N be a normal π'-subgroup of π-soluble group G. Then $G \in \mathfrak{X}(\pi)$ if and only if $G/N \in \mathfrak{X}(\pi)$.*

Proof. (1) Suppose that $G \in \mathfrak{X}(\pi)$ and $N \lhd G$. Let M/N be a maximal subgroup with π-number index in G/N. Then M is a maximal subgroup with π-number index in G, and so $|\pi(M)| < |\pi(G)|$. If $r \in \pi(G) \setminus \pi(M)$, then

$$r \notin \pi(N),\ r \in \pi(G/N) \setminus \pi(M/N).$$

Hence $|\pi(M/N)| < |\pi(G/N)|$ and $G/N \in \mathfrak{X}(\pi)$. Thus, $\mathfrak{X}(\pi)$ is a homomorph.

Now we show that $\mathfrak{X}(\pi)$ is a a saturated class. Assume that $G/\Phi(G)\in\mathfrak{X}(\pi)$ and let M be a maximal subgroup with π-number index in G. Then $M/\Phi(G)$ is a maximal subgroup with π-number index in $G/\Phi(G)$. By hypothesis, $|\pi(M/\Phi(G))|<|\pi(G/\Phi(G))|$. In view of [1, Theorem 4.33], $\pi(G/\Phi(G))=\pi(G)$. Therefore

$$|\pi(M)|=|\pi(M/\Phi(G))|<|\pi(G/\Phi(G))|=|\pi(G)|,\ G\in\mathfrak{X}(\pi).$$

(2) Suppose that $G\in\mathfrak{X}(\pi)$. Then G is π-soluble, and in view of 1.1 (1) G is π_1-soluble. If M is a maximal subgroup of G such that $\pi(G:M)\subseteq\pi_1\subseteq\pi$, then $|\pi(M)|<|\pi(G)|$ since $G\in\mathfrak{X}(\pi)$. Consequently, $G\in\mathfrak{X}(\pi_1)$ and $\mathfrak{X}(\pi)\subseteq\mathfrak{X}(\pi_1)$.

(3) Assume that $G\in\mathfrak{X}(\pi(G))$. Then G is soluble and has no wide subgroups, that is, for every maximal subgroup M of G we have

$$\pi(G:M)\subseteq\pi(G)\subseteq\pi,\ |\pi(M)|<|\pi(G)|.$$

Hence $G\in\mathfrak{X}(\pi)$.

(4) If $G\in\mathfrak{X}(\pi)$, then by assertion (1) $G/N\in\mathfrak{X}(\pi)$ for any normal subgroup N of G.

Conversely, let N be a normal π'-subgroup of a π-soluble group G. Suppose that $G\notin\mathfrak{X}(\pi)$. Then $\pi(A)=\pi(G)$ for some maximal subgroup A of G such that $\pi(G:A)\subseteq\pi$. Since N is a π'-group, we have $N\subseteq A$. Now, A/N is a maximal subgroup of G/N and

$$|G:A|=|G/N:A/N|,\ \pi(G/N:A/N)\subseteq\pi.$$

By inductive hypothesis, $G/N\in\mathfrak{X}(\pi)$, consequently, $|\pi(A/N)|\neq|\pi(G/N)|$. Assume that

$$r\in\pi(G/N)\setminus\pi(A/N).$$

Then $r\in\pi(G)=\pi(A)$. Since $r\in\pi(A)$ and $r\notin\pi(A/N)$, it follows that some Sylow r-subgroup A_r of A is contained in N. As N is a π'-subgroup, therefore $r\in\pi'$. Since $\pi(G:A)\subseteq\pi$, we see that A_r is a Sylow r-subgroup of G and $r\notin\pi(G/N)$, a contradiction. Thus we conclude that $G\in X(\pi)$.

Lemma is proved.

***Example* 2.2.** Let p and q de different primes, n be the least positive integer such that q devides p^n-1. There exists $S=[E_{p^n}]Q$, where E_{p^n} is an elementary abelian subgroup of order p^n, $|Q|=q$. In S all proper subgroups are primary. Therefore $S\in\mathfrak{X}(\{p,q\})$. It is clear that a cyclic group Z_p of order p belongs to $\mathfrak{X}(\{p,q\})$. A group $G=S\times Z_q$ contains a wide subgroup $E_{p^n}\times Z_q$. Hence $G\notin\mathfrak{X}(\{p,q\})$. Since a formation is closed under direct products, we obtain that $\mathfrak{X}(\{p,q\})$ is not a formation.

Further note that if a maximal subgroup M of G contains a Sylow p-subgroup, then M is normal in G. Hence the order of the factor group G/M_G is equal to q and $G/M_G\in\mathfrak{X}(\{p,q\})$. If a maximal subgroup H of G contains a Sylow q-subgroup, then H is not normal in G and coincides with a Sylow q-subgroup. Therefore $H=Q^g\times Z_q$, $g\in S$, and $H_G=Z_q$. Hence $G/H_G\simeq S$ and $G/H_G\in\mathfrak{X}(\{p,q\})$. It follows that all primitive factor groups of G belong to $\mathfrak{X}(\{p,q\})$, but $G\notin\mathfrak{X}(\{p,q\})$. Thus $\mathfrak{X}(\{p,q\})$ is not a Schunck class.

***Lemma* 2.3.**

(1) *If* $G\in\mathfrak{X}(\pi)$, *then* $\Phi(G)$ *is a* π'*-group.*

(2) $G\in\mathfrak{X}(\pi)$ *if and only if* $G/\Phi(G)\in X(\pi)$.

(3) *If* $G\in X(\pi)$ *and* $O_{\pi'}(G)=1$, *then* $F(G)$ *is a Hall subgroup and every Sylow subgroup of* $F(G)$ *is a minimal normal subgroup in G.*

Proof. (1) Suppose that $p\in\pi(\Phi(G))\cap\pi$. By Lemma 1.3, in G there exists a maximal subgroup M such that $|G:M|=p^a$. Note that M is a Hall subgroup in G since $G\in\mathfrak{X}(\pi)$, and so $p\notin\pi(M)$. But $p\in\pi(\Phi(G))\subseteq\pi(M)$. This contradiction shows that $\Phi(G)$ is a π'-group.

(2) Assume that $G\in\mathfrak{X}(\pi)$. Then by assertion (1), $\Phi(G)$ is a π'-group. Consequently, by Lemma 2.1 (4), $G/\Phi(G)\in\mathfrak{X}(\pi)$.

Conversely, let $G/\Phi(G)\in\mathfrak{X}(\pi)$. Hence, in view of Lemma 2.1 (1), $G\in\mathfrak{X}(\pi)$.

(3) By assertion (1), $\Phi(G)=1$. Let N be a minimal normal subgroup of G. Then N is a p-subgroup for some $p\in\pi\cap\pi(G)$. By [1, Theorem 3.20], there exists a subgroup M such that $G=[N]M$. Note that M is a maximal subgroup and $\pi(G:M)=\{p\}$. By hypothesis, $\pi(M)\neq\pi(G)$, therefore M is a p'-Hall subgroup and N is a Sylow p-subgroup of G.

Lemma is proved.

***Lemma* 2.4.** *A soluble group G is quasi-k-primary if and only if* $G\in\mathfrak{X}(\pi(G))$ *and* $|\pi(G)|=k+1$.

Proof. Suppose that a soluble group G is quasi-k-primary. Then G has no wide maximal subgroups and $G\in\mathfrak{X}(\pi(G))$. We show that $|\pi(G)|=k+1$. Since G is quasi-k-primary, it follows that for every maximal subgroup H of G we have $|\pi(H)|\leq k<|\pi(G)|$, and there exists a maximal subgroup M such that $|\pi(M)|=k$. In view of [1, Theorem 4.14], maximal subgroups of a soluble group have primary indices, and so $|G:M|=p^\alpha$, $p\in\pi(G)$, $\alpha\in\mathbb{N}$. Since $|G|=|M|\cdot|G:M|$, we have $|\pi(G)|\leq k+1$. Thus, $|\pi(G)|=k+1$.

Conversely, assume that $G \in \mathfrak{X}(\pi(G))$ and $|\pi(G)| = k+1$. Then G is $\pi(G)$-soluble, and so G is soluble. Also, G has no wide maximal subgroups, i. e., for every maximal subgroup M of G we have $|\pi(M)| < |\pi(G)| = k+1$. Hence $|\pi(M)| = k$ and G is quasi-*k*-primary. Lemma is proved.

3 The structure of groups from the class $\mathfrak{X}(\pi)$

In π-soluble groups indices of maximal subgroups are primes from π or π'-numbers. It follows that if a group belongs to the class $X(\pi)$, then its every maximal subgroup with π-number index is a Hall subgroup. Such groups are described by V.S. Monakhov [3].

***Lemma* 3.1.** [3]. *Let G be a π-soluble group. The following assertions are equivalent.*

(1) *Chief π-factors of G are isomorphic to Sylow subgroups.*

(2) *Every maximal subgroup with π-number index is a Hall subgroup.*

(3) *The set of all maximal subgroups with π-number indices of G coincides with the set of all p-supplements for all $p \in \pi$.*

(4) *A Hall π-subgroup of every normal πd-subgroup of G is a π-Hall subgroup of G.*

***Theorem* 3.2.** *Let G be a π-soluble group, $\pi \cap \pi(G) \neq \varnothing$ and $O_\pi(G) \neq 1$. Then $G \in \mathfrak{X}(\pi)$ if and only if $G = [G_p]M$, where G_p is a minimal normal Sylow p-subgroup of G for some $p \in \pi(O_\pi(G))$, M is a maximal subgroup of G and $M \in \mathfrak{X}(\pi)$.*

Proof. Assume that $G \in \mathfrak{X}(\pi)$. If $H \lessdot G$ such that $\pi(G:H) \subseteq \pi$, then by Lemma 1.2 (2), $|G:H| = r^\alpha$ for some $r \in \pi$ and positive integer α. Since $\pi(G) \neq \pi(H)$, it follows that $r \notin \pi(H)$ and H is a r'-Hall subgroup. Therefore G satisfies assertion (2) of Lemma 3.1. By Lemma 1.2 (1), there exists a minimal normal p-subgroup N for some $p \in \pi(O_\pi(G))$ since $O_\pi(G) \neq 1$. In view of Lemma 3.1 (4), *N* is a Sylow *p*-subgroup of *G*, i. e., $N = G_p$. By Lemma 1.3, there exists a maximal subgroup *M* of G such that $|G:M| = p^a$, $a \in \mathbb{N}$. Hence $G = [G_p]M$, and so by Lemma 2.1 (1), $M \in \mathfrak{X}(\pi)$.

Conversely, suppose that $G = [G_p]M$, where G_p is a minimal normal Sylow *p*-subgroup of G for some $p \in \pi(O_\pi(G))$, M is a maximal subgroup of G and $M \in \mathfrak{X}(\pi)$. Let K be a maximal subgroup of G with π-number index. Then by Lemma 1.2 (2), $|G:K| = r^\alpha$ for some $r \in \pi$ and positive integer α. If $r = p$, then $G = G_pK$. Hence $G_p \cap K = 1$ and $M \simeq K$, and so $|\pi(K)| = |\pi(M)| < |\pi(G)|$. If $r \neq p$, then $G_p < K$. Therefore $K = [G_p]M \cap K = [G_p](M \cap K)$, and so $|\pi(K)| < |\pi(G)|$. Otherwise

$$\pi(M) = \pi(G) \setminus \{p\} = \pi(K) \setminus \{p\} = \pi(M \cap K),$$

and this is a contradiction since $M \in \mathfrak{X}(\pi)$. Thus, $G \in \mathfrak{X}(\pi)$. Theorem is proved.

***Corollary* 3.2.1.** *Let G be a π-soluble group and $\pi \cap \pi(G) \neq \varnothing$. Then $G \in \mathfrak{X}(\pi)$ if and only if*

$$G / O_{\pi'}(G) = [G_pO_{\pi'}(G) / O_{\pi'}(G)]M / O_{\pi'}(G),$$

where $G_pO_{\pi'}(G) / O_{\pi'}(G)$ is a minimal normal Sylow p-subgroup of $G / O_{\pi'}(G)$, $p \in \pi(O_\pi(G / O_{\pi'}(G)))$, M is a maximal subgroup of G and $M \in \mathfrak{X}(\pi)$.

Proof. If $O_{\pi'}(G) = 1$, then in view of Theorem 3.2 corollary is true.

Let $O_{\pi'}(G) \neq 1$. Then by Lemma 1.2 (3),

$$O_\pi(G / O_{\pi'}(G)) \neq 1.$$

Suppose that $G \in \mathfrak{X}(\pi)$. In view of Lemma 2.1 (4), $G / O_{\pi'}(G) \in \mathfrak{X}(\pi)$. By Theorem 3.2,

$$G / O_{\pi'}(G) = [G_pO_{\pi'}(G) / O_{\pi'}(G)]M / O_{\pi'}(G),$$

where $G_pO_{\pi'}(G) / O_{\pi'}(G)$ is a minimal normal Sylow *p*-subgroup of $G / O_{\pi'}(G)$, $p \in \pi(O_\pi(G / O_{\pi'}(G)))$, and $M / O_{\pi'}(G)$ is a maximal subgroup of $G / O_{\pi'}(G)$, $M / O_{\pi'}(G) \in \mathfrak{X}(\pi)$. Hence M is a maximal subgroup of G and $M \in \mathfrak{X}(\pi)$ in view of Lemma 2.1 (4).

Conversely, assume that *G* can be represented as

$$G / O_{\pi'}(G) = [G_pO_{\pi'}(G) / O_{\pi'}(G)]M / O_{\pi'}(G),$$

where $G_pO_{\pi'}(G) / O_{\pi'}(G)$ is a minimal normal Sylow *p*-subgroup of $G / O_{\pi'}(G)$, $p \in \pi(O_\pi(G / O_{\pi'}(G)))$, M is a maximal subgroup of G and $M \in \mathfrak{X}(\pi)$. Then $M / O_{\pi'}(G)$ is a maximal subgroup of $G / O_{\pi'}(G)$ and $M / O_{\pi'}(G) \in \mathfrak{X}(\pi)$ in view of Lemma 2.1 (4). Consequently, by Theorem 3.2, $G / O_{\pi'}(G) \in \mathfrak{X}(\pi)$. Hence $G \in \mathfrak{X}(\pi)$ by Lemma 2.1 (4). Corollary is proved.

***Corollary* 3.2.2.** *A soluble group G is quasi-k-primary if and only if $G = [G_p]M$, where G_p is a minimal normal Sylow p-subgroup of G for some $p \in \pi(G)$, M is a maximal quasi-$(k-1)$-primary subgroup of G.*

Proof. Suppose that a soluble group *G* is quasi-*k*-primary. Then in view of Lemma 2.4, $G \in \mathfrak{X}(\pi(G))$ and $|\pi(G)| = k+1$. By Theorem 3.2, $G = [N]M$, where N is a minimal normal Sylow *p*-subgroup of G for some $p \in \pi(G)$, M is a maximal subgroup of G and $M \in \mathfrak{X}(\pi(G))$. Hence by Lemma 2.1 (2), $M \in \mathfrak{X}(\pi(M))$. Besides, $|\pi(M)| = |\pi(G)| - |\pi(N)| = k$. Consequently, M is quasi-$(k-1)$-primary by Lemma 2.4.

Conversely, assume that a soluble group G can be represented as $G=[N]M$, where N is a minimal normal Sylow p-subgroup of G for some $p\in\pi(G)$, M is a maximal quasi-$(k-1)$-primary subgroup of G. Then by Lemma 2.4, $M\in\mathfrak{X}(\pi(M))$ and $|\pi(M)|=k$. In view of Lemma 2.1 (3), $M\in\mathfrak{X}(\pi(G))$. Since $|\pi(G)|=|\pi(M)|+|\pi(N)|=k+1$, it follows that G is quasi-k-primary by Lemma 2.4. Corollary is proved.

If we substitute $k=2$ in Corollary 3.2.2, then we obtain the result of S. S. Levischenko.

***Corollary* 3.2.3** [9, Theorem 3.1]. *A soluble quasibiprimary group G is equal to the semidirect product* $[P]M$ *of its elementary abelian Sylow p-subgroup P and quasiprimary subgroup M, which is also a maximal subgroup of G.*

A group G is said to be π-special, if $G=G_\pi\times G_{\pi'}$ and G_π is nilpotent.

Let G be a nontrivial group,

$$Z_0(G)=1,\ Z_1(G)=Z(G),$$
$$Z_2(G)/Z_1(G)=Z(G/Z_1(G)),\ \ldots,$$
$$Z_i(G)/Z_{i-1}(G)=Z(G/Z_{i-1}(G)),\ \ldots.$$

Then the subgroup $Z_\infty(G)=\bigcup_{i=0}^{\infty}Z_i(G)$ is called the hypercenter of G.

Obviously, $Z(G/Z_\infty(G))=1$.

***Theorem* 3.3.** *If every wide maximal subgroup of a* π*-soluble group G with* π*-primary index is* π*-special, then* $G/Z_\infty(G)\in\mathfrak{X}(\pi)$.

Proof. Let $G/Z_\infty(G)\notin\mathfrak{X}(\pi)$, and write $\overline{G}=G/Z_\infty(G)$. Then in $\overline{G}$ there exists a maximal subgroup $\overline{M}=M/Z_\infty(G)$ such that $|\overline{G}:\overline{M}|\subseteq\pi$ and $\pi(\overline{M})=\pi(\overline{G})$. At the same time M is maximal in G and $\pi(M)=\pi(G)$. By hypothesis, M is π-special. And so $\overline{M}$ is also π-special, that is, $\overline{M}=\overline{M}_\pi\times\overline{M}_{\pi'}$ and $\overline{M}_\pi$ is nilpotent. By Lemma 1.2 (2), $|\overline{G}:\overline{M}|=p^a$ for some $p\in\pi$ and positive integer a. It follows that $\overline{M}_p\lhd\overline{M}$ and $\overline{M}_p$ is a proper subgroup of $\overline{G}_p$. Therefore $\overline{M}_p$ is normal in $\overline{G}$. Thus there exists a nontrivial element $\overline{x}\in\overline{M}_p\cap Z(\overline{G}_p)$ such that it belongs to the center of $\overline{G}$, a contradiction since $Z(G/Z_\infty(G))=1$. Theorem is proved.

***Corollary* 3.3.1.** *If every wide maximal subgroup of a soluble group G is nilpotent, then* $G/Z_\infty(G)$ *is quasi-k-primary, where* $k=|\pi(G/Z_\infty(G))|-1$.

Proof. Suppose that every wide maximal subgroup of a soluble group G is nilpotent. Then, substituting $\pi=\pi(G)$ in Theorem 3.3, we obtain $G/Z_\infty(G)\in\mathfrak{X}(\pi(G))$. Hence by Lemma 2.4, $G/Z_\infty(G)$ is quasi-k-primary, where

$$k=|\pi(G/Z_\infty(G))|-1.$$

Corollary is proved.

A group G is π-decomposable if $G=G_\pi\times G_{\pi'}$.

***Corollary* 3.3.2.** *If every maximal subgroup of a* π*-soluble group G is normal and* π*-decomposable, then* G_π *is nilpotent and* $G/Z_\infty(G)\in\mathfrak{X}(\pi)$.

Proof. Since every maximal subgroup of a π-soluble group G is normal, it follows that $G=G_\pi[G_{\pi'}]$ and G_π is nilpotent by Lemma 1.6. Hence since every maximal subgroup of G is π-decomposable, it is π-special, and $G/Z_\infty(G)\in\mathfrak{X}(\pi)$ by Theorem 3.3. Corollary is proved.

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