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SHORT COMMUNICATIONS

On Boundary Value Problems for First-Order Elliptic Pseudosymmetric Systems in $\mathbf{R}^4$

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The investigation of well-posedness of a boundary value problem for a given system of partial differential equations plays an important role in the classification of elliptic systems. By [1–4], the regularizability[1] or nonregularizability of specific boundary value problems for some elliptic systems depends on whether the system belongs to a specific homotopic class. There are also systems [6–8] for which the regularizability of specific boundary value problems has nothing or little to do with the homotopy type of the system and is determined by other factors. Finally, there exist elliptic systems [9, 10] for which there are no boundary conditions that give a regularizable boundary value problem. The number of known systems of the above-mentioned type is small, and we increase this number in the present paper.

It was proved in [11] that, for an arbitrary pseudosymmetric system of first-order differential equations in $\mathbf{R}^4$, any boundary value problem of the type of the Riemann–Hilbert problem cannot be Fredholm. It turns out (as will be proved below) that systems singled out by Vinogradov are systems of the type of [9, 10], i.e., arbitrary boundary conditions for them cannot give a regularizable boundary value problem. In particular, this answers the question posed in [12] for a particular system of differential equations (see system (12)[2] in [12]).

In a bounded domain $\Omega \subset \mathbf{R}^4$ whose boundary is a sufficiently smooth three-dimensional manifold $\partial\Omega$, we consider the boundary value problem of finding a solution $U = U(x)$ of the elliptic system of differential equations

$$\sum_{j=1}^{4} A_j \frac{\partial U}{\partial x_j} = f(x), \qquad x \in \Omega, \tag{1}$$

satisfying the boundary conditions

$$\mathscr{B}(y, \partial/\partial x)|_{x\to y} U = g(y), \qquad y \in \partial\Omega. \tag{2}$$

Here the A_j $(j = 1, 2, 3, 4)$ are constant real matrices of the fourth order; moreover, A_1 is the identity matrix, and the remaining matrices are skew-symmetric (i.e., $A_j^{\mathrm{T}} = -A_j$ for $j = 2, 3, 4$); U and f are four-component column vectors, g is a two-component column vector, and $\mathscr{B}$ is a 2×4 matrix boundary operator consisting of scalar linear sufficiently smooth pseudodifferential operators "polynomial" in the normal to $\partial\Omega$ [13].

Theorem. *For an arbitrary boundary operator* $\mathscr{B}$, *the boundary value problem* (1), (2) *is not regularizable.*

In particular, it follows from the theorem that the operator corresponding to the boundary value problem (1), (2) and acting in certain Banach spaces [13, 14] is not Fredholm. In other words, this operator has an infinite-dimensional kernel or cokernel. For example, if $\mathscr{B}$ is the

[1] We say that a boundary value problem is regularizable if the Lopatinskii condition is valid for it [5].

[2] This system contained misprints: the signs should be replaced by the opposite ones for the term $d_2 w_t$ in the second equation of the system, u_t in the third one, and $d_2 u_t$ in the fourth equation, and the term $-e_2 u_x$ in the third equation should be replaced by $-c_2 u_x$.

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operator of multiplication of the vector U by a constant 2×4 matrix, then the kernel of the operator corresponding to problem (1), (2) is infinite-dimensional [11].

Proof of the theorem. It suffices to prove that the Lopatinskii condition does not hold for the boundary value problem in the half-space $\mathbf{R}^4_+ := \{x=(x_1,x_2,x_3,x_4)\in\mathbf{R}^4 \mid x_1>0\}$ obtained from problem (1), (2) by freezing the coefficients at a point of $\partial\Omega$, where the normal to $\partial\Omega$ is parallel to the axis Ox_1. Therefore, throughout the following, we assume that $\Omega=\mathbf{R}^4_+$ and the symbol of the boundary operator $\mathscr{B}$ occurring in (2) is independent of the point $y\in\partial\mathbf{R}^4_+$.

Without loss of generality, one can assume that the boundary conditions (2) do not contain the differentiation with respect to x_1. Let $A(\lambda,\tau) := A_1\lambda + A_2\tau_1 + A_3\tau_2 + A_4\tau_3$ be the characteristic matrix of system (1). By $B(\tau)$ we denote the symbol of the principal part of the operator $\mathscr{B}(-i\,\partial/\partial x)|_{\partial\Omega}$. To justify the theorem, one must show that, for at least one nonzero triple $\tau=(\tau_1,\tau_2,\tau_3)\in\mathbf{R}^3\backslash\{0\}$, the rank of the matrix

$$B(\tau)\int\limits_{\gamma} A^{-1}(\lambda,\tau)d\lambda \tag{3}$$

is strictly less than 2 [here γ is a simple smooth closed contour lying in the upper λ-half-plane and surrounding all roots of the equation $\det A(\lambda,\tau)=0$ lying there].

The characteristic matrix $A(\lambda,\tau)$ of system (1) has the form

$$A(\lambda,\tau)=\begin{pmatrix} \lambda & a(\tau) & b(\tau) & c(\tau)\\ -a(\tau) & \lambda & r(\tau) & -q(\tau)\\ -b(\tau) & -r(\tau) & \lambda & p(\tau)\\ -c(\tau) & q(\tau) & -p(\tau) & \lambda \end{pmatrix}, \tag{4}$$

where $a(\tau),b(\tau),\ldots,r(\tau)$ are linear forms in the variables τ_1, τ_2, and τ_3 with real coefficients. Since system (1) is elliptic, it follows from [15, Lemmas 1 and 2] that the quadratic form

$$d(\tau) := a(\tau)p(\tau)+b(\tau)q(\tau)+c(\tau)r(\tau)$$

is sign-definite and each of the systems $\{a(\tau),b(\tau),c(\tau)\}$ and $\{p(\tau),q(\tau),r(\tau)\}$ of linear forms is linearly independent. To be definite, we assume that $d(\tau)>0$ for $\tau\in\mathbf{R}^3\backslash\{0\}$; the case of a negative-definite quadratic form $d(\tau)$ can be considered in a similar way.

The nondegenerate change of variables $(\tau_1',\tau_2',\tau_3')=(a(\tau),b(\tau),c(\tau))$ reduces the matrix (4) to a matrix of the same form with $a(\tau)=\tau_1$, $b(\tau)=\tau_2$, and $c(\tau)=\tau_3$.

The matrix $B(\tau)$ is a 2×4 matrix whose entries are real continuous homogeneous functions of the variables τ_1, τ_2, and τ_3. We assume that its rank is equal to 2 at each point $\tau\in\mathbf{R}^3\backslash\{0\}$. [Otherwise, the rank of the matrix (3) is necessarily less than 2 for some $\tau\neq 0$.]

By Λ_{jk} and H_{jk} $(k,j=1,2,3,4)$ we denote the second-order minors formed by the jth and kth columns of the matrix $B(\tau)$ and the matrix (3), respectively. We set $L_1(\tau)=\Lambda_{12}+\Lambda_{34}$, $L_2(\tau)=\Lambda_{13}+\Lambda_{42}$, $L_3(\tau)=\Lambda_{14}+\Lambda_{23}$, and $\Delta(\tau)=\left(p^2(\tau)+q^2(\tau)+r^2(\tau)+2d+\tau_1^2+\tau_2^2+\tau_3^2\right)^{1/2}$. A straightforward computation of the minors H_{jk} $(1\le j<k\le 4)$ of the matrix (3) shows that $H_{23}=H_{14}$, $H_{24}=-H_{13}$, $H_{34}=H_{12}$, and, neglecting a constant nonzero factor,

$$\begin{aligned} d^{-1}H_{12} &= \left((q+\tau_2)^2+(r+\tau_3)^2\right)L_1(\tau)-(p+\tau_1)(q+\tau_2)L_2(\tau)\\ &\quad -(p+\tau_1)(r+\tau_3)L_3(\tau)+i\Delta(\tau)\left(-(r+\tau_3)L_2(\tau)+(q+\tau_2)L_3(\tau)\right),\\ d^{-1}H_{13} &= -(p+\tau_1)(q+\tau_2)L_1(\tau)+\left((p+\tau_1)^2+(r+\tau_3)^2\right)L_2(\tau)\\ &\quad -(q+\tau_2)(r+\tau_3)L_3(\tau)+i\Delta(\tau)\left((r+\tau_3)L_1(\tau)-(p+\tau_1)L_3(\tau)\right),\\ d^{-1}H_{14} &= -(p+\tau_1)(r+\tau_3)L_1(\tau)-(q+\tau_2)(r+\tau_3)L_2(\tau)+\left((p+\tau_1)^2+(q+\tau_2)^2\right)L_3(\tau)\\ &\quad +i\Delta(\tau)\left(-(q+\tau_2)L_1(\tau)+(p+\tau_1)L_2(\tau)\right). \end{aligned}$$

Since $p(\tau)$, $q(\tau)$, and $r(\tau)$ are linearly independent forms and the quadratic form $d(\tau)$ is positive definite, it follows that the forms $p(\tau)+\tau_1$, $q(\tau)+\tau_2$, and $r(\tau)+\tau_3$ are linearly independent. Consequently, the linear mapping $\varphi : \mathbf{R}^3 \to \mathbf{R}^3$ given by $\varphi(\tau) = (p(\tau)+\tau_1, q(\tau)+\tau_2, r(\tau)+\tau_3)$ is nondegenerate. Therefore, its restriction to the closed unit ball $B_1[0] := \{\tau \in \mathbf{R}^3 \mid |\tau| \le 1\}$ is a homeomorphism of this ball onto a convex body $T \subset \mathbf{R}^3$ for which $0 \in \mathbf{R}^3$ is an interior point. Hence, in turn, we find that the mapping ψ of the boundary ∂T of T onto the unit sphere S^2 in $\mathbf{R}^3$ given by $\psi(\zeta) = \zeta/|\zeta|$ is a homeomorphism of ∂T onto S^2.

Since the rank of the matrix $B(\tau)$ is equal to 2 for each $\tau \ne 0$, it follows that the vector $L(\tau) = (L_1(\tau), L_2(\tau), L_3(\tau))$ is nonzero at each point $\tau \in \mathbf{R}^3\backslash\{0\}$. Consequently, the continuous nondegenerate vector field[3] $L \circ \varphi^{-1} \circ \psi^{-1}$ is defined on the two-dimensional sphere S^2. By the well-known hedgehog lemma, there exists a point $\xi_0 \in S^2$ such that $L \circ \varphi^{-1} \circ \psi^{-1}(\xi_0) = \alpha \cdot \xi_0$ for some nonzero number $\alpha \in \mathbf{R}$. The last relation implies that

$$L_1(\tau_0) = \alpha\left(p(\tau_0) + \tau_1^{(0)}\right)\Big/|\varphi(\tau_0)|, \qquad L_2(\tau_0) = \alpha\left(q(\tau_0) + \tau_2^{(0)}\right)\Big/|\varphi(\tau_0)|,$$

$$L_3(\tau_0) = \alpha\left(r(\tau_0) + \tau_3^{(0)}\right)\Big/|\varphi(\tau_0)|$$

at the point $\tau_0 = \varphi^{-1} \circ \psi^{-1}(\xi_0) \in S^2$. By substituting the resulting expressions for $L_1(\tau_0)$, $L_2(\tau_0)$, and $L_3(\tau_0)$ into the minors H_{jk} of the matrix (3), we obtain $H_{jk}(\tau_0) = 0$ $(k, j = 1, 2, 3, 4)$. Therefore, the rank of the matrix (3) is less than 2 at the point $\tau_0 = \varphi^{-1} \circ \psi^{-1}(\xi_0) \ne 0$. The proof of the theorem is complete.

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[3] For brevity, the mapping $(\varphi|_{S^2})^{-1}$ is denoted by φ^{-1}.