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Solvable groups with restrictions on Sylow subgroups of the Fitting subgroup

Alexander Trofimuk

Department of Mathematics, Brest State University named after A. S. Pushkin, 21, Boulevard of Cosmonauts Brest, Belarus
alexander.trofimuk@gmail.com

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In this paper, we study solvable groups in which $r_n(F)$ is at most 2. In particular, we investigated groups of odd order and A_4-free groups with this property. Exact estimations of the derived length and nilpotent length of such groups are obtained.

Keywords: Fitting subgroup; normal rank; Sylow subgroups; A_4-free groups; derived length; nilpotent length.

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1. Introduction

All groups considered in this paper will be finite. All notations and definitions correspond to [4].

The structure of a solvable group depends primarily on its Fitting subgroup. The following Baer's result is well-known, see [4, p. 720]:

Let G be a finite solvable group. If

$$\Phi(G) = N_0 \subset N_1 \subset \cdots \subset N_{m-1} \subset N_m = F(G), \tag{1}$$

is a normal series such that $N_i \lhd G$ and N_i/N_{i-1} has a prime order, $i = 1, 2, \ldots, m$, *then G is supersolvable.* Here, $\Phi(G)$ is the Frattini subgroup of G, $F(G)$ is the Fitting subgroup of G.

Recall that a group is bicyclic if it is the product of two cyclic subgroups.

In work [8] notice that the estimation of the derived length depends only on the Sylow subgroups of the Fitting subgroup. The following assertion was proved:

Let G be a solvable non-primary group and $F(G)$ is its Fitting subgroup. If all Sylow subgroups of $F(G)$ are bicyclic, then the derived length of G is at most 6.

Monakhov [5] introduced the concept of the normal rank $r_n(P)$ of p-group P as follows:

$$r_n(P) = \max_{X \lhd P} \log_p |X/\Phi(X)|, \tag{2}$$

where X runs over all normal subgroups of P including P. The basis theorem of Burnside [4, III.3.15] implies that the normal rank $r_n(P)$ is the least natural number k such that every normal subgroup of a p-group P is generated by at most k elements.

It is obvious that p-group is cyclic, if and only if its normal rank is equal to 1.

The normal rank of bicyclic p-group is at least 3. So in Huppert's paper [3] there is the 2-group

$$G = \langle a, b, c \,|\, a^2 = b^8 = c^2 = 1, [a, b] = c, [b, c] = b^4, [a, c] = 1\rangle.$$

It is bicyclic group of order 2^5 and $r_n(G) = 3$. From [4, III.11.5] follows, that the normal rank of bicyclic p-group, $p \neq 2$ is at most 2. However, the converse is not true. So if S is extraspecial of order 27, then $r_n(S) = 2$, but S is not bicyclic. Besides, every 2-group of the normal rank ≤ 2 is bicyclic by Lemma 2.4.

The structure of solvable groups with Sylow subgroups of normal rank ≤ 2 was obtained by Monakhov in [5]. In particular, the following statement was proved:

If G is solvable with Sylow subgroups of normal rank ≤ 2, then the nilpotent length of G is at most 4.

To simplify the presentation, we introduce the following notation:

$$r_n(F) = \max_{p \in \pi(F)} r_n(F_p).$$

Here F is the Fitting subgroup of G, F_p is a Sylow p-subgroup of F for some prime $p \in \pi(F)$. The set of all prime divisors of $|F|$ is denoted $\pi(F)$.

In this paper, we study solvable groups in which $r_n(F)$ is at most 2. We proved the following theorem.

Theorem 1.1. *Let G be a solvable group and $r_n(F) \leq 2$. Then the nilpotent length of G is at most 4 and the derived length of G is at most 6. In particular, if:*

(1) *G is A_4-free, then the nilpotent length of G is at most 3 and the derived length of G is at most 4;*
(2) *G has odd order, then G is metanilpotent and the derived length of G is at most 3.*

Recall that a group is metanilpotent if it has a nilpotent normal subgroup such that the quotient group is also a nilpotent group. We say that G is A_4-free if there is no section isomorphic to the alternating group A_4 of degree 4.

We write $[A]B$ for a semidirect product with a normal subgroup A.

Example 1.1. Let S be a extraspecial group of order 27. The calculations in the computer system GAP [2] show that the automorphism group of S is $[E_{3^2}]GL(2, 3)$, where E_{3^2} is an elementary Abelian group of order 3^2. The semidirect product $G = [S]GL(2,3)$ is a solvable group of order $1296 = 2^4 3^4$ with the Fitting subgroup $F = S$ and $r_n(F) = 2$. The nilpotent length of G equals 4, the derived length of G equals 6. Hence the estimations of the nilpotent length and the derived length, which are obtained in general case of Theorem 1.1, are exact.

Example 1.2. Let A be a extraspecial group of order 125. The semidirect product $G = [A]S_3$ is A_4-free of order $750 = 5^3 \cdot 3 \cdot 2$ with the Fitting subgroup $F = A$ and $r_n(F) = 2$. Here S_3 is the symmetric group of degree 3. The nilpotent length of G equals 3, the derived length of G equals 4. Hence the estimations of the nilpotent length and the derived length, which are obtained in Theorem 1.1 for A_4-free groups, are exact.

A non-nilpotent group whose proper subgroups are all nilpotent is called a Schmidt group.

Example 1.3. Fix a prime number $p = 5$ and $q = 3$. Since the order of 5 modulo 3 is equal to 2, there is a Schmidt group $G = [P]Q$ such that P is a non-Abelian subgroup of order 5^3, Q is a cyclic subgroup of order 3. In particular, the Fitting subgroup $F = P$ and $r_n(F) = 2$. Since P is non-Abelian, $Z(P) = P' = \Phi(P)$. By the properties of Schmidt groups, we have $G' = P$. Thus $((G')')' = (P')' = (Z(P))' = 1$ and the derived length of G equals 3. Obviously that the nilpotent length of G is equal to 2. Hence the estimations of the nilpotent length and the derived length, which are obtained in Theorem 1.1 for groups of odd order, are exact.

2. Preliminary Results

Let $\mathfrak{F}$ and $\mathfrak{H}$ be non-empty formations. If G is a group then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual of G, that is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$. We define $\mathfrak{F} \circ \mathfrak{H} = \{G \mid G^{\mathfrak{H}} \in \mathfrak{F}\}$ and call $\mathfrak{F} \circ \mathfrak{H}$ the formation product of $\mathfrak{F}$ and $\mathfrak{H}$, see [1, IV, 1.7]. As usually, $\mathfrak{F}^2 = \mathfrak{F} \circ \mathfrak{F}$ and $\mathfrak{F}^n = \mathfrak{F}^{n-1} \circ \mathfrak{F}$ for every natural $n \geq 3$. A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. In this paper, $\mathfrak{N}$ and $\mathfrak{A}$ denote the formations of all nilpotent and all Abelian groups, respectively. The other definitions and terminology about formations could be referred to [7].

To prove the main theorem, we need the following lemmas.

Lemma 2.1. *Let $\mathfrak{F}$ be a formation. Then $\mathfrak{N} \circ \mathfrak{F}$ is saturated formation.*

Proof. By [7, p. 36], the product $\mathfrak{N} \circ \mathfrak{F}$ is local formation. Since the concepts of "saturated formation" and "local formation" are equivalent, then $\mathfrak{N} \circ \mathfrak{F}$ is saturated formation. □

In the Huppert's monograph a description of p-groups G in which every Abelian normal subgroup generated by no more than two elements was obtained. These results are shown in Lemmas 2.2 and 2.3.

Lemma 2.2 ([4, Theorem III.7.6]). *Let G be a p-group and every Abelian normal subgroup be cyclic. Then:*

(1) *if $p > 2$, then G is cyclic*;
(2) *if $p = 2$, then P has a normal cyclic subgroup of index 2.*

Lemma 2.3 ([4, Theorem III.12.4, Remark III.12.5]). *Let G be a p-group, $|G| = p^n$ and every Abelian normal subgroup has two generators. Then G is one of the following groups*:

(I) *If $p \geq 3$, then*:
- (I_1) *G is metacyclic*;
- (I_2) *either $G = A \times B$, where A is non-Abelian group of order p^3 and exponent p, B is cyclic of order p^{n-2}, or $G = [A]B$, where $A = Z_p \times Z_{p^{n-2}}$ is Abelian, B is cyclic of order p*;
- (I_3) *$G = [A]B$, where A is Abelian, $A = C_G(G')$, B is cyclic of order p*;
- (I_4) *G is a 3-group of maximal class.*

(II) *If $p = 2$, then*:
- (II_1) *G is the quaternion group of order 8*;
- (II_2) *G is a central product of two subgroups Q_8 and D_8, where D_8 is the dihedral group of order 8*;
- (II_3) *G is a special group such that $|G/Z(G)| = 2^4$ and $|Z(G)| = 2^2$.*

Lemma 2.4. *Let P be a p-group and $r_n(P) \leq 2$. Then the derived length of P is at most 2. In particular, if $p = 2$, then P is bicyclic.*

Proof. Since $r_n(P) \leq 2$, then every Abelian normal subgroup has no more than two generators. If every Abelian normal subgroup is cyclic, then by Lemma 2.2, we have that P is bicyclic and the derived length of P is at most 2. For the case when the number of generators of each Abelian normal subgroup is equal to 2, we use Lemma 2.3. Obviously that the groups from (I_1), (I_3) and (II_1) are metabelian. Since non-Abelian group A of order p^3 and exponent p is metabelian, it follows that P from (I_2) is metabelian. From (I_4) the derived length of 3-group of maximal class equals 2 by [4, III.14.17]. The order of group P from (II_2) is equal to 16 and the number of P in the library SmallGroups [2] is 8. Moreover, this group is bicyclic and has the derived length equal to 2. The calculations in the computer system GAP show that the group from (II_3) has the normal rank equal to 4. Therefore, it is excluded from consideration.

Thus, the derived length of P is at most 2. Moreover, if $p = 2$, then P is bicyclic. $\square$

Lemma 2.5 ([6, Lemma 12]). *Let H be an irreducible solvable subgroup of $GL(2, p)$. Then $H \in \mathfrak{N}^3 \cap \mathfrak{A}^4$.*

Lemma 2.6 ([6, Lemma 13]). *If H is a solvable A_4-free subgroup of $GL(2, p)$, then H is metabelian.*

Lemma 2.7 ([4, Lemma VI.8.1]). *Let H be an irreducible subgroup of $GL(2,p)$ and H has odd order. Then H is cyclic.*

3. Proof of Theorem 1.1

(1) We first show that $G \in \mathfrak{F} = \mathfrak{N}^4 \cap \mathfrak{N} \circ \mathfrak{A}^4$. Apply induction on $|G|$. Assume that $\Phi(G) \neq 1$. Hence $F(G/\Phi(G)) = F(G)/\Phi(G)$. Let F_p be a Sylow p-subgroup of $F = F(G)$. Then $F_p\Phi(G)/\Phi(G)$ is a Sylow p-subgroup in $F(G/\Phi(G))$. Since $F_p\Phi(G)/\Phi(G) \cong F_p/F_p \cap \Phi(G)$, it follows that $r_n(F_p\Phi(G)/\Phi(G)) \leq r_n(F_p) \leq 2$ and $r_n(F(G/\Phi(G))) \leq r_n(F) \leq 2$. Hence $G/\Phi(G)$ satisfies the hypothesis of the theorem. Since $\mathfrak{F}$ is a saturated formation, $G \in \mathfrak{F}$. Next we assume that $\Phi(G) = 1$.

By [4, III.4.5], F is the direct product of minimal normal subgroups N_i of G, where $1 \leq i \leq k$. By [4, I.4.5], for any N_i the quotient group $G/C_G(N_i)$ is isomorphic to an irreducible subgroup of $\mathrm{Aut}(N_i)$. By [4, I.9.6], the quotient group $G/\bigcap_{i=1}^k C_G(N_i)$ is isomorphic to a subgroup of the direct product of $G/C_G(N_i)$, $1 \leq i \leq k$. Since the Fitting subgroup of any solvable group G with $\Phi(G) = 1$ contains its centralizer, it follows that

$$\bigcap_{i=1}^k C_G(N_i) = C_G(F) = F \quad \text{and} \quad G\Big/\bigcap_{i=1}^k C_G(N_i) = G/F.$$

Since N_i is an elementary Abelian p_i-subgroup of order p_i^k, it follows that $N_i \leq F_{p_i}$ and $k \leq 2$, seeing $\Phi(N_i) = 1$ and $r_n(P) \leq 2$ for any Sylow subgroup P from $F(G)$. Therefore, the following options are possible:

(1) $\mathrm{Aut}(N_i)$ is isomorphic to cyclic group of order $p_i - 1$;

(2) $\mathrm{Aut}(N_i)$ isomorphic to $GL(2,p_i)$;

In the first case, $G/C_G(N_i)$ is cyclic. Hence $G/C_G(N_i) \in \mathfrak{A} \subseteq \mathfrak{N}^3 \cap \mathfrak{A}^4$.

In the second case, $G/C_G(N_i)$ is an irreducible subgroup of $GL(2,p_i)$ and $G/C_G(N_i) \in \mathfrak{N}^3 \cap \mathfrak{A}^4$ by Lemma 2.5. Since $\mathfrak{N}^3 \cap \mathfrak{A}^4$ is a formation, it follows that $G/F \in \mathfrak{N}^3 \cap \mathfrak{A}^4$. Hence $G \in \mathfrak{F}$.

From all the above, we proved that $G/F \in \mathfrak{A}^4$. By Lemma 2.4, $F \in \mathfrak{A}^2$. Therefore, the derived length of G is at most 6. Since $G \in \mathfrak{N}^4$, it follows that the nilpotent length of G is at most 4.

Let G be an A_4-free. Repeating the proof of the main part of theorem and using Lemma 2.6, we obtained that $G/F \in \mathfrak{A}^2$. Then $G \in \mathfrak{N}^3$ and the nilpotent length of G is at most 3. Since $F \in \mathfrak{A}^2$, the derived length of G is at most 4 by Lemma 2.4.

Let G has odd order. By Lemma 2.7, $G/F \in \mathfrak{A}$. Then $G \in \mathfrak{N}^2$ and the nilpotent length of G is at most 2 and the derived length of G is at most 3 by Lemma 2.4.

The theorem is proved.

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