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# SUPERSOLUBILITY OF A FINITE GROUP WITH NORMALLY EMBEDDED MAXIMAL SUBGROUPS IN SYLOW SUBGROUPS

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**Abstract:** Let $P$ be a subgroup of a Sylow subgroup of a finite group $G$. If $P$ is a Sylow subgroup of some normal subgroup of $G$ then $P$ is called *normally embedded* in $G$. We establish tests for a finite group $G$ to be $p$-supersoluble provided that every maximal subgroup of a Sylow $p$-subgroup of $X$ is normally embedded in $G$. We study the cases when $X$ is a normal subgroup of $G$, $X = O_{p',p}(H)$, and $X = F^{\star}(H)$ where $H$ is a normal subgroup of $G$.

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## 1. Introduction

We consider only finite groups. All notations and definitions below correspond to [1, 2].

In 1980, in [3], Srinivasan proved the supersolubility of a group whose all maximal subgroups of each Sylow subgroup are normal in the group. These groups were called MNP-*groups* (see [4]). In what follows, groups with constraints on the maximal subgroups of Sylow subgroups were studied in many works (see, for example, [5–10]).

A subgroup $H$ of a group $G$ is called *normally embedded* in $G$ if for each Sylow subgroup $P$ in $H$ there exists a normal subgroup $K$ in $G$ such that $P$ is a Sylow subgroup of $K$ (see [11, Definition I.7.1]). The notation $H\ ne\ G$ means a subgroup $H$ normally embedded in a group $G$. Investigations of the structure of a group with normally embedded subgroups are reflected in [11].

If a $p$-subgroup $P$ of a group $G$ is normally embedded in $G$ then $P$ is a Sylow $p$-subgroup of $P^G$. Here $P^G$ is the normal closure of the subgroup $P$ in $G$, i.e., the least normal subgroup of $G$ containing $P$. The study of the structure of groups whose some subgroups are Hall subgroups of their normal closures is presented in [12–15].

It is clear that every normal $p$-subgroup is normally embedded. Therefore, each MNP-group is a group whose every maximal subgroup of each Sylow subgroup is normally embedded in the group. The converse fails. In the dihedral group

$$D_{12} = \langle a, b \mid a^6 = b^2 = 1,\ a^b = a^5 \rangle$$

of order 12, all maximal subgroups of the Sylow subgroups are normally embedded in $D_{12}$ but $\langle b \rangle$ is not normal in $D_{12}$.

In this article, we establish tests for a group $G$ to be $p$-supersoluble provided that every maximal subgroup of a Sylow $p$-subgroup of $X$ is normally embedded in $G$. We examine the cases of $X = H$ and $X = O_{p',p}(H)$, where $H$ is a normal subgroup of $G$. In particular, we prove

**Theorem.** *Suppose that $H$ is a normal subgroup of a $p$-soluble group $G$ and $G/H$ is $p$-supersoluble.*

(1) *If all maximal subgroups of a Sylow $p$-subgroup of $H$ are normally embedded in $G$ then $G$ is $p$-supersoluble.*

(2) *If all maximal subgroups of a Sylow $p$-subgroup of $O_{p',p}(H)$ are normally embedded in $G$ then $G$ is $p$-supersoluble.*

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In the case when $p$ is the least prime divisor of the order of the group, it is possible to reject the condition of $p$-solubility in item (1) of the theorem (see Corollary 3.1). In item (2), the condition of $p$-solubility cannot be omitted. An example is given by $G = H = \mathrm{SL}(2,5)$ in which $\mathrm{O}_{2',2}(H) = Z(H)$, $|Z(H)| = 2$, but $\mathrm{SL}(2,5)$ is not 2-supersoluble.

From the theorem, we deduce some new tests for the supersolubility of a group (see Corollaries 3.1–3.4 and 4.1).

The theorem is proved in Sections 3 and 4 (see Theorems 3.1 and 4.1). Applications of the obtained results for the groups $G$ with constraints on the maximal subgroups of Sylow subgroups of $F^*(G)$ are presented in Section 5.

## 2. Auxiliary Results

Denote by $Z(G)$, $F(G)$, $F^\star(G)$, and $\Phi(G)$ the center, the Fitting subgroup, the generalized Fitting subgroup, and the Frattini subgroup of a group $G$ respectively; $Z_m$ and $E_{p^t}$ are the cyclic and elementary abelian groups of orders $m$ and $p^t$ respectively; $\mathrm{O}_p(G)$ and $\mathrm{O}_{p'}(G)$ are the greatest $p$- and $p'$-subgroups of a group $G$ respectively, $\mathrm{O}_{p',p}(G)$ is the greatest normal $p$-nilpotent subgroup of $G$, and $\pi(G)$ is the set of all prime divisors of the order of $G$. If $|\pi(G)| = 1$ then the group $G$ is called *primary*, and if $|\pi(G)| = 2$ then it is called *biprimary.*

A *normal series* of $G$ is a chain of subgroups

$$1 = G_0 \leq G_1 \leq \dots \leq G_m = G,$$

in which the subgroup $G_i$ is normal in $G$ for all $i = 0, 1, \dots, m$. The quotient groups $G_{i+1}/G_i$ are called the *factors* of this series.

A group $G$ *has a Sylow tower* if $G$ admits a normal series whose factors are isomorphic to Sylow subgroups. If

$$\pi(G) = \{p_1, p_2, \dots, p_n\}, \quad p_1 > p_2 > \dots > p_m,$$

and for each $i$ the factor $G_i/G_{i-1}$ is isomorphic to a Sylow $p_i$-subgroup of $G$ then the group $G$ has a *Sylow tower of supersoluble type.*

Each supersoluble group has a Sylow tower of supersoluble type [1, Theorem VI.9.1]. The alternating group $A_4$ of degree 4 has a Sylow tower of nonsupersoluble type.

Let $p$ be a prime. A group is called *$p$-soluble* if the orders of its chief factors are either a degree of $p$ or do not divide by $p$. A group is called *$p$-supersoluble* if the orders of its chief factors are either equal to $p$ or do not divide by $p$. Denote by $p\mathfrak{S}$ the class of all $p$-soluble groups and designate as $p\mathfrak{U}$ the class of all $p$-supersoluble groups. A group with a normal Sylow $p$-subgroup is called *$p$-closed*, and a group with normal $p'$-Hall subgroup is called *$p$-nilpotent.* The semidirect product of a normal subgroup $A$ in $G$ and a subgroup $B$ is written as follows: $G = [A]B$.

**Lemma 2.1** [11, Lemma I.7.3]. *Let $U$ be a normally embedded $p$-subgroup of a group $G$ and let $K$ be a normal subgroup of $G$. Then the following hold:*

(1) *if $U \leq H \leq G$ then $U$ ne $H$;*

(2) *$UK/K$ ne $G/K$;*

(3) *$U \cap K$ ne $G$;*

(4) *if $K$ is a $p$-group then $UK$ ne $G$ and $U \cap K$ is normal in $G$;*

(5) *$U^g$ ne $H$ for every $g \in G$.*

**Lemma 2.2.** *Suppose that $H$ is a normal subgroup of a group $G$ and every maximal subgroup of a Sylow $p$-subgroup of $H$ is normally embedded in $G$. If $N$ is a normal subgroup of $G$ then each maximal subgroup of every Sylow $p$-subgroup of $HN/N$ is normally embedded in $G/N$.*

Proof. Lemma 2.1(5) implies that, for every Sylow $p$-subgroup $X$ in $H$ and every maximal subgroup $X_1$ in $X$, the subgroup $X_1$ is normally embedded in $G$. Let $\overline{P_1} = X/N$ be a maximal subgroup of a Sylow $p$-subgroup $\overline{P}$ in $HN/N$. Then $N \leq X \leq HN$ and there exists a Sylow $p$-subgroup $P$ in $HN$

such that $\overline{P} = PN/N$. By [1, Theorem VI.4.6], there exist Sylow $p$-subgroups $H_p$ in $H$ and $N_p$ in $N$ such that $P = H_pN_p$; therefore, $\overline{P} = H_pN/N$. Further,

$$N \leq X < PN \leq H_pN, \quad X = (X \cap H_p)N.$$

Since $H_p \cap N = X \cap H_p \cap N$, we have

$$p = |\overline{P} : \overline{P_1}| = |H_pN/N : X/N| = |H_pN : X|$$
$$= |H_pN : (X \cap H_p)N| = \frac{|H_p||N||X \cap H_p \cap N|}{|H_p \cap N||X \cap H_p||N|} = |H_p : X \cap H_p|.$$

Hence, $X \cap H_p$ is a maximal subgroup of $H_p$. By hypothesis, $X \cap H_p$ is normally embedded in $G$. By Lemma 2.1(2), $(X \cap H_p)N/N = X/N$ is normally embedded in $G/N$.

**Lemma 2.3.** (1) *If $K$ is a normal subgroup of a group $G$ and $H$ is a normal subgroup of $K$ then $O_{p',p}(K) \cap H = O_{p',p}(H)$.*

(2) *If $H$ is a normal subgroup of $G$ then $O_{p',p}(HO_{p'}(G)) \cap H = O_{p',p}(H)$.*

(3) *If $H$ is a normal subgroup of $G$ then $O_{p',p}(H\Phi(G)) \cap H = O_{p',p}(H)$.*

Proof. (1) Since $O_{p',p}(K) \cap H$ is a normal $p$-nilpotent subgroup of $H$ then $O_{p',p}(K) \cap H \leq O_{p',p}(H)$. The subgroup $O_{p',p}(H)$ is normal in $K$ and $p$-nilpotent; therefore, $O_{p',p}(H) \leq O_{p',p}(K) \cap H$. Consequently, $O_{p',p}(K) \cap H = O_{p',p}(H)$.

(2) The assertion stems from item (1) for $K = HO_{p'}(G)$.

(3) The assertion follows from item (1) for $K = H\Phi(G)$.

It is well known that the class $p\mathfrak{U}$ is a saturated hereditary formation.

**Lemma 2.4** [16, Lemma 5]. *Suppose that a $p$-soluble group $G$ does not belong to $p\mathfrak{U}$ but $G/K \in p\mathfrak{U}$ for every nontrivial normal subgroup $K$ in $G$. Then the following hold:*

(1) *$Z(G) = O_{p'}(G) = \Phi(G) = 1$;*

(2) *$G$ contains a unique minimal normal subgroup $N$; i.e., $N = F(G) = O_p(G) = C_G(N)$;*

(3) *$G$ is a primitive group; $G = [N]M$, where $M$ is a maximal subgroup of $G$ with trivial kernel;*

(4) *$N$ is an elementary abelian subgroup of order $p^n$, $n > 1$;*

(5) *if a subgroup $M$ is abelian then $M$ is cyclic of an order dividing $p^n - 1$ and $n$ is the least natural such that $p^n \equiv 1 \pmod{|M|}$.*

## 3. Supersolubility of a Group with Normally Embedded Maximal Subgroups of Sylow Subgroups

Denote by $\mathfrak{X}_p$ the class consisting of all groups $G$ for which each maximal subgroup of a Sylow $p$-subgroup of $G$ is normally embedded in $G$. Lemma 2.1(5) implies that, for every Sylow $p$-subgroup $T$ in $G$ and every maximal subgroup $T_1$ in $T$, the subgroup $T_1$ is normally embedded in $G$. Let $\mathfrak{X} = \bigcap_{p\in\mathbb{P}} \mathfrak{X}_p$, where $\mathbb{P}$ is the set of all primes. Then $\mathfrak{X}$ consists of all groups for which every maximal subgroup of every Sylow subgroup of $G$ is normally embedded in $G$.

**Lemma 3.1.** *If $p$ is a prime and $G \in \mathfrak{X}_p$ then the following hold:*

(1) *if $N$ is a normal subgroup of $G$ then $G/N \in \mathfrak{X}_p$;*

(2) *if $V$ is a subgroup of $G$ and $p$ does not divide $|G : V|$ then $V \in \mathfrak{X}_p$.*

Proof. 1. Let $PN/N$ be a Sylow $p$-subgroup of $G/N$, where $P$ is a Sylow $p$-subgroup of $G$ and let $M/N$ be a maximal subgroup of $PN/N$. Then there exists a subgroup $P_1$ in $P$ such that $M = P_1N$. Obviously, $P_1$ is a maximal subgroup of $P$. By hypothesis, $P_1$ is a Sylow $p$-subgroup of $P_1^G$ and $P_1N/N$ is a Sylow $p$-subgroup of $(P_1N/N)^{G/N} = P_1^GN/N$. Indeed,

$$\big|P_1^GN/N : P_1N/N\big| = \big|P_1^GN : P_1N\big| = \big|P_1^G : P_1\big|/\big|P_1^G \cap N : P_1 \cap N\big|.$$

Since $\left|P_1^G : P_1\right|$ does not divide by $p$, $|P_1^G N/N : P_1 N/N|$ does not divide by $p$. Hence, $G/N \in \mathfrak{X}_p$.

2. Let $P_1$ be a maximal subgroup of a Sylow $p$-subgroup $P$ of $V$. By hypothesis, $p$ does not divide $|G : V|$; therefore, $P$ is a Sylow $p$-subgroup of $G$. Since $G \in \mathfrak{X}_p$, the subgroup $P_1$ is a Sylow subgroup of $P_1^G$. The subgroup $P_1^G \cap V$ is normal in $V$; therefore, $P_1^V \leq P_1^G \cap V$. Consequently, $P_1$ is a Sylow subgroup of $P_1^V$ and $V \in \mathfrak{X}_p$.

EXAMPLE 3.1. Non-Hall subgroups of $G \in \mathfrak{X}$ may fail to belong to $\mathfrak{X}$. This is confirmed by the group $G = [S]Z_2$, where $S$ is an extraspecial group of order 27, and $Z_2$ is a cyclic group of order 2. This group has number IdGroup$(G) = [54, 8]$ in the SmallGroups library of the computer algebra system GAP [17]. The group $G = [S]Z_2$ belongs to $\mathfrak{X}$ but the subgroup $H = Z_3 \times S_3$ does not belong to $\mathfrak{X}$. Here $S_3$ is the symmetric group of degree 3.

EXAMPLE 3.2. The class $\mathfrak{X}$ is not closed under direct products. For example, the group $S_3$ belongs to $\mathfrak{X}$ but the direct product $G = A \times B$ does not belong to $\mathfrak{X}$, where $A \cong B \cong S_3$. Indeed, let

$$A = [\langle a\rangle]\langle b\rangle, \quad a^3 = b^2 = 1, \quad a^b = a^2,$$

$$B = [\langle c\rangle]\langle d\rangle, \quad c^3 = d^2 = 1, \quad c^d = c^2.$$

The subgroup $P = \langle a\rangle \times \langle c\rangle$ is a normal Sylow 3-subgroup of $G$. The subgroup $M = \langle ac^2\rangle$ is maximal in $P$ but $M$ is not normal in $G$ since $(ac^2)^b = a^2c^2 \notin M$. Hence, $M$ is not Sylow in its normal closure $M^G = P$.

**Lemma 3.2.** *Let $\{P_i\}$, $i = 1, 2, \dots, m$, be the set of all maximal subgroups in a Sylow $p$-subgroup $P$ of a group $G$. If $G \in \mathfrak{X}_p$ then the subgroup $K = \bigcap_{i=1}^m P_i^G$ is $p$-nilpotent.*

PROOF. Let $R = K \cap P_1$ be a Sylow $p$-subgroup of $K$. Suppose that $R \not\leq P_j$ for some $j > 1$. Then

$$RP_j = P \leq KP_j^G = P_j^G;$$

a contradiction to the fact that $P_j$ is a Sylow subgroup of $P_j^G$. Therefore,

$$R \leq \bigcap_{i=1}^{m} P_i = \Phi(P). \tag{3.1}$$

By [1, Theorem IV.4.7], $K$ is a $p$-nilpotent group.

**Lemma 3.3.** *If $G \in \mathfrak{X}_p$ and $p$ is the least element in $\pi(G)$ then $G$ is $p$-nilpotent.*

PROOF. Induct on the order of the group. Let $P$ be a Sylow $p$-subgroup of $G$. If $|P| = p$ then $G$ is $p$-nilpotent (see [1, Theorem IV.2.8]). Assume from now on that $|P| > p$. Let $P_i$, $i = 1, 2, \dots, m$, be all maximal subgroups of $P$. Since $P_i \neq 1$, $i = 1, 2, \dots, m$, and $G/P_i^G \in \mathfrak{X}_p$ by Lemma 3.1, by induction, the quotient group $G/P_i^G$ is $p$-nilpotent for all $i$. If $K = \bigcap_{i=1}^m P_i^G = 1$ then $G$ is $p$-nilpotent. Let $K \neq 1$. By Lemma 3.2, $K$ is a $p$-nilpotent subgroup. Therefore, the $p'$-Hall subgroup $K_{p'}$ is normal in $G$. If $K_{p'} \neq 1$ then by induction $G/K_{p'}$ is $p$-nilpotent, and hence $G$ is $p$-nilpotent. Let $K$ be a nontrivial $p$-group. Since $G/K$ is $p$-nilpotent, $G_{p'}K$ is normal in $G$. Considering (3.1) and [1, Theorem IV.4.7], we conclude that $G_{p'}K = G_{p'} \times K$. Therefore, $G$ is $p$-nilpotent.

**Lemma 3.4.** *If $G \in \mathfrak{X}$ then $G$ has a Sylow tower of supersoluble type.*

PROOF. Induct on the order of the group. Let $p$ be the least prime in $\pi(G)$. Since $G \in \mathfrak{X}_p$, by Lemma 3.3, $G$ is $p$-nilpotent. By induction, a $p'$-Hall subgroup of $G$ has a Sylow tower of supersoluble type. Consequently, $G$ has a Sylow tower of supersoluble type.

**Theorem 3.1.** *Suppose that $H$ is a normal subgroup of a $p$-soluble group $G$ and the quotient group $G/H$ is $p$-supersoluble. If every maximal subgroup of a Sylow $p$-subgroup of $H$ is normally embedded in $G$ then $G$ is $p$-supersoluble.*

PROOF. If $H = 1$ then the theorem holds. Let $H \neq 1$ and let $N$ be a normal subgroup of $G$. The subgroup $HN/N$ is normal in $G/N$ and

$$(G/N)/(HN/N) \cong G/(HN) \cong (G/H)/(HN/H)$$

is $p$-supersoluble. Lemma 2.2 enables us to apply induction to the quotient group $G/N$; therefore, $G/N$ is $p$-supersoluble. Now we can use Lemma 2.4, by which $Z(G) = O_{p'}(G) = \Phi(G) = 1$, the group $G$ contains a unique minimal normal subgroup

$$N = F(G) = O_p(G) = C_G(N), \quad G = [N]M.$$

Hence, $N$ is an elementary abelian subgroup of order $p^n$, $n > 1$, and $M$ is a maximal subgroup of $G$. Since $N \leq H$, the subgroup $N$ is included in every Sylow $p$-subgroup $P$ in $H$.

Suppose that $N = P$. By hypothesis, each maximal subgroup $S$ in $P$ is normally embedded in $G$. Then, by Lemma 2.1(4), $S = S \cap P$ is normal in $G$; a contradiction to the fact that $N$ is a minimal normal subgroup of $G$.

From now on, we assume that $N < P$. By Dedekind's identity,

$$P = P \cap [N]M = [N](P \cap M), \quad 1 \neq P \cap M < P.$$

Choose a maximal subgroup $T$ in $P$ such that $P \cap M \leq T$. By hypothesis, $T$ is a Sylow subgroup of $T^G$. Since $1 \neq T \leq T^G$, we have $N \leq T^G$. Now, $P = NT \leq T^G$; a contradiction. The theorem is proved.

Show that the condition of the $p$-solubility of $G$ can be omitted in Theorem 3.1 a priori if $p$ is the least element in $\pi(H)$.

**Corollary 3.1.** *Suppose that $H$ is a normal subgroup of a group $G$, $p$ is the least element in $\pi(H)$, and the quotient group $G/H$ is $p$-supersoluble. If each maximal subgroup of a Sylow $p$-subgroup of $H$ is normally embedded in $G$ then $G$ is $p$-supersoluble.*

PROOF. By Lemma 2.1(1), $H \in \mathfrak{X}_p$. By Lemma 3.3, $H$ is $p$-nilpotent. Now, $G$ is $p$-soluble. It remains to apply Theorem 3.1.

**Corollary 3.2.** *Suppose that $H$ is a normal subgroup of a group $G$ and $G/H$ is supersoluble. If each maximal subgroup of every Sylow subgroup of $H$ is normally embedded in $G$ then $G$ is supersoluble. In particular, if every maximal subgroup of every Sylow subgroup of $G$ is normally embedded in $G$ then $G$ is supersoluble.*

PROOF. Applying Corollary 3.1 and Theorem 3.1, we conclude that $G$ is $p$-supersoluble for all $p \in \pi(H)$. If $r \in \pi(G) \setminus \pi(H)$ then $H$ is an $r'$-subgroup. Since $G/H$ is supersoluble, $G$ is $r$-supersoluble. Hence, $G$ is $p$-supersoluble for $p \in \pi(G)$; therefore, $G$ is supersoluble. For $G = H$, we infer that if each maximal subgroup of every Sylow subgroup of $G$ is normally embedded in $G$ then $G$ is supersoluble.

EXAMPLE 3.3. The group $G = Z_3 \times S_3$ [17, IdGroup$(G) = [18, 3]$] is supersoluble but $G \notin \mathfrak{X}$ since not all maximal subgroups of a normal Sylow 3-subgroup are normal in $G$. Therefore, the class of supersoluble groups is wider than $\mathfrak{X}$.

EXAMPLE 3.4. The group $D_{12}$ belongs to $\mathfrak{X}$ but not all maximal subgroups of a Sylow 2-subgroup are normal in $G$. Therefore, the class $\mathfrak{X}$ is wider than the class of MNP-groups.

**Corollary 3.3.** *In a biprimary group $G$, each maximal subgroup of every Sylow subgroup of $G$ is normally embedded in $G$ if and only if $G$ is supersoluble and each maximal subgroup of a normal Sylow subgroup is normal in $G$.*

PROOF. Let $G = QP$ be a biprimary group, where $P$ is a Sylow $p$-subgroup, $Q$ is a Sylow $q$-subgroup, and $p < q$. Suppose that every maximal subgroup of every Sylow subgroup is normally embedded in $G$. By Corollary 3.2, $G$ is supersoluble; therefore, $Q$ is normal in $G$. If $Q_1$ is a maximal subgroup of $Q$ then, by Lemma 2.1(4), $Q_1 = Q_1 \cap Q$ is normal in $G$. Necessity is proved.

Prove sufficiency. Let $G = [Q]P$ be a biprimary supersoluble group, where $P$ is a Sylow $p$-subgroup, $Q$ is a Sylow $q$-subgroup, $p < q$, and let every maximal subgroup of $Q$ be normal in $G$. If $P_1$ is a maximal subgroup of $P$ then $QP_1$ is normal in $G$; therefore, $P_1 \leq P_1^G \leq QP_1$. Since $P_1$ is a Sylow subgroup of $QP_1$, the subgroup $P_1$ is a Sylow subgroup in $P_1^G$. Consequently, each maximal subgroup of every Sylow subgroup in $G$ is normally embedded in $G$.

Example 3.5. If $G = [P]Q$ is a biprimary group and each element in $Q$ performs a power automorphism on $P$ then $G \in \mathfrak{X}$.

Example 3.6. In $G = S_3 \times ([Z_7]Z_3)$ [17, IdGroup$(G) = 126, 8$], the only normal Sylow subgroup is the Sylow 7-subgroup $P$. The Sylow 2- and 3-subgroups of $G$ perform power automorphisms on $P$. But $G \notin \mathfrak{X}$. Therefore, the biprimarity condition in Corollary 3.3 cannot be omitted.

**Corollary 3.4.** *Suppose that $G$ is a group and $p$ is the greatest element in $\pi(G)$. If for each $r \in \pi(G) \setminus \{p\}$ each maximal subgroup of every Sylow $r$-subgroup is normally embedded in $G$ then the following hold:*

(1) *$G$ has a Sylow tower of supersoluble type;*

(2) *$G/O_p(G)$ is supersoluble.*

Proof. Let $q \in \pi(G)$, where $q$ is the least element. By Lemma 3.3, $G$ is $q$-nilpotent. Let $U$ be a $q'$-Hall subgroup of $G$. Note that $U$ is normal in $G$. By Lemma 3.1, all conditions of the corollary hold for $U$. By induction, $U$ has a Sylow tower of supersoluble type; therefore, $G$ has a Sylow tower of supersoluble type. Consequently, $O_p(G)$ is a Sylow $p$-subgroup of $G$. By Lemma 3.1, the quotient group $G/O_p(G)$ belongs to $\mathfrak{X}$, and by Corollary 3.2, $G/O_p(G)$ is supersoluble.

## 4. Supersolubility of a Group with Normally Embedded Subgroups in $O_{p',p}(H)$

**Lemma 4.1.** *If $H$ is a normal subgroup of a group $G$ then*

$$O_p(HO_{p'}(G)/O_{p'}(G)) = O_{p',p}(H)O_{p'}(G)/O_{p'}(G). \tag{4.1}$$

Proof. Let $P$ be a Sylow $p$-subgroup of $O_{p',p}(H)$. Then $O_{p',p}(H) = [O_{p'}(H)]P$ and the subgroup

$$O_{p',p}(H)O_{p'}(G) = [O_{p'}(G)]P$$

is normal in $HO_{p'}(G)$. Therefore,

$$O_{p',p}(H)O_{p'}(G)/O_{p'}(G) = [O_{p'}(G)]P/O_{p'}(G) \leq O_p(HO_{p'}(G)/O_{p'}(G)). \tag{4.2}$$

Since $HO_{p'}(G)/O_{p'}(G) \simeq H/H \cap O_{p'}(G) = H/O_{p'}(H)$, we have

$$O_p(HO_{p'}(G)/O_{p'}(G)) \simeq O_p(H/O_{p'}(H)) = O_{p',p}(H)/O_{p'}(H). \tag{4.3}$$

Further,

$$O_{p',p}(H)O_{p'}(G)/O_{p'}(G) \simeq O_{p',p}(H)/O_{p',p}(H) \cap O_{p'}(G) = O_{p',p}(H)/O_{p'}(H). \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$|O_p(HO_{p'}(G)/O_{p'}(G))| = |O_{p',p}(H)O_{p'}(G)/O_{p'}(G)|. \tag{4.5}$$

Now, inclusion (4.2) and equality (4.5) imply equality (4.1).

**Lemma 4.2.** *If $H$ is a normal subgroup of $G$ and $O_{p'}(G) = 1$ then*

$$O_p(H\Phi(G)/\Phi(G)) = O_p(H)\Phi(G)/\Phi(G).$$

Proof. If $\Phi(G)=1$ then the desired equality holds. Let $\Phi(G)\neq 1$. Since $O_{p'}(G)=1$, we infer that $\Phi(G)$ is a $p$-subgroup. Therefore,

$$O_p(H)\Phi(G)/\Phi(G)\leq O_p(H\Phi(G)/\Phi(G)).$$

Let $P/\Phi(G)=O_p(H\Phi(G)/\Phi(G))$. Then $P\leq H\Phi(G)$ and $P=(H\cap P)\Phi(G)$. Since $H\cap P$ is normal in $H$, $H\cap P\leq O_p(H)$ and

$$O_p(H\Phi(G)/\Phi(G))=P/\Phi(G)=(H\cap P)\Phi(G)/\Phi(G)\leq O_p(H)\Phi(G)/\Phi(G).$$

**Theorem 4.1.** *Suppose that $H$ is a normal subgroup of a $p$-soluble group $G$ and $G/H$ is $p$-supersoluble. If each maximal subgroup of a Sylow $p$-subgroup of $O_{p',p}(H)$ is normally embedded in $G$ then $G$ is $p$-supersoluble.*

Proof. Lemma 2.1(5) implies that, for each Sylow $p$-subgroup $T$ in $O_{p',p}(H)$ and each maximal subgroup $T_1$ in $T$, the subgroup $T_1$ is normally embedded in $G$. Apply induction on $|G|+|H|$. Suppose that $O_{p'}(G)\neq 1$. Check that the quotient group $G/O_{p'}(G)$ and the normal subgroup $HO_{p'}(G)/O_{p'}(G)$ satisfy all hypotheses of the theorem. The group

$$(G/O_{p'}(G))/(HO_{p'}(G)/O_{p'}(G))\cong G/(HO_{p'}(G))\cong (G/H)/(HO_{p'}(G)/H)$$

is $p$-supersoluble. Since $O_{p'}(G/O_{p'}(G))=1$, we have

$$O_{p'}(HO_{p'}(G)/O_{p'}(G))=1.$$

By Lemma 4.1,

$$O_{p',p}(HO_{p'}(G)/O_{p'}(G))=O_p(HO_{p'}(G)/O_{p'}(G))=O_{p',p}(H)O_{p'}(G)/O_{p'}(G).$$

Since $O_{p',p}(H)$ is normal in $G$, by Lemma 2.2, each maximal subgroup of a Sylow $p$-subgroup in $O_{p',p}(HO_{p'}(G)/O_{p'}(G))$ is normally embedded in $G/O_{p'}(G)$. By induction, $G/O_{p'}(G)$ is $p$-supersoluble, and so $G$ is $p$-supersoluble. Thus, from now on we assume that $O_{p'}(G)=1$.

It is similarly checked on using Lemma 4.2 that $\Phi(G)=1$. Thus, we may assume that

$$O_{p'}(G)=\Phi(G)=O_{p'}(H)=\Phi(H)=1.$$

Therefore, for the Fitting subgroup $F(G)$ we have

$$F(G)=N_1\times N_2\times\cdots\times N_k, \tag{4.6}$$

where $N_i$ are elementary abelian minimal normal $p$-subgroups of $G$.

Let $N$ be an arbitrary minimal normal $p$-subgroup of $G$. If $N\leq H$ then $N\leq O_p(H)$. Since $\Phi(O_p(H))=1$, there exists a maximal subgroup $S$ in $O_p(H)$ with $NS=O_p(H)$. Since $O_p(H)$ is normal in $G$, by Lemma 2.1(4), $S=S\cap O_p(H)$ is normal in $G$. Obviously, $N\cap S=1$ and $|N|=|O_p(H):S|=p$. If $N\not\leq H$ then $N\cap H=1$. Since $NH/H$ is a minimal normal subgroup of $G/H$ and by hypothesis the quotient group $G/H$ is $p$-supersoluble, $|N|=|NH/H|=p$. Thus, the order of every minimal normal subgroup $N_i$ in decomposition (4.6) is equal to the prime $p$. Therefore, for each subgroup $N_i$, the quotient group $G/C_G(N_i)$ is isomorphic to a subgroup of a cyclic group of order $p-1$. Moreover, $G/\bigcap_{i=1}^k C_G(N_i)$ is isomorphic to a subgroup of the direct product of $G/C_G(N_i)$, $1\leq i\leq k$. Since, in a $p$-soluble group with $\Phi(G)=O_{p'}(G)=1$, the Fitting subgroup coincides with its centralizer [18, Lemma 2 (2)], we infer

$$\bigcap_{i=1}^k C_G(N_i)=C_G(F(G))=F(G),\quad G/\bigcap_{i=1}^k C_G(N_i)=G/F(G)$$

and $G/F(G)$ is an abelian $p'$-group. Therefore, $G$ is a soluble group and $G$ is supersoluble by Baer's Theorem [1, p. 720]. The theorem is proved.

For $G = H$ we obtain

**Corollary 4.1.** *Let $G$ be a $p$-soluble group. If each maximal subgroup of a Sylow $p$-subgroup of $O_{p',p}(G)$ is normally embedded in $G$ then $G$ is $p$-supersoluble.*

## 5. Supersolubility of a Group with Normally Embedded Maximal Subgroups of the Sylow Subgroups of $F^\star(G)$

A subgroup $H$ of a group $G$ is called *c-normal* in $G$ (see [19]) if there exists a normal subgroup $K$ in $G$ such that $HK = G$ and $H \cap K \leq H_G$, where $H_G$ is the greatest normal subgroup of $G$ contained in $H$.

The groups with $c$-normal maximal subgroups of Sylow subgroups were studied in [20, 21]. In particular, we have

**Lemma 5.1** [21, Theorem 3.1]. *Suppose that $H$ is a normal subgroup of a group $G$ and the quotient group $G/H$ is supersoluble. If every maximal subgroup of every Sylow subgroup of $F^\star(H)$ is $c$-normal in $G$, then $G$ is supersoluble.*

Example 5.1. Let $G$ be the holomorph of the cyclic group $Z_5$ of order 5. Then $G = [Z_5]Z_4$ and maximal subgroups of Sylow subgroups have orders 1 and 2. The subgroup $Z_2$ of order 2 is a Sylow subgroup of the normal subgroup $[Z_5]Z_2$; therefore, all maximal subgroups of the Sylow subgroups of $G$ are normally embedded in $G$. But $Z_2$ is not $c$-normal. Consequently, normally embedded maximal subgroups in Sylow subgroups may fail to be $c$-normal.

But if the Sylow subgroups are taken from $F^\star(H)$ (see [22, X.13]) then the normal embeddedness of the maximal subgroups implies their normality and, hence, their $c$-normality. This is proved in the following

**Theorem 5.1.** *Suppose that $H$ is a normal subgroup of a group $G$ and $G/H$ is supersoluble. If every maximal subgroup of every Sylow subgroup of $F^\star(H)$ is normally embedded in $G$ then*
(1) *$F^\star(H) = F(H)$;*
(2) *each maximal subgroup of every Sylow subgroup of $F^\star(H)$ is normal in $G$;*
(3) *$G$ is supersoluble.*

Proof. (1) Let $P$ be a Sylow $p$-subgroup of $F^\star(H)$ and let $P_1, \dots, P_m$ be maximal subgroups of $P$. By hypothesis, each subgroup $P_i$ is normally embedded in $G$. By Lemma 2.1(1), $P_i$ is normally embedded in $F^\star(H)$. Since $P$ is an arbitrary Sylow subgroup of $F^\star(H)$, we have $F^\star(H) \in \mathfrak{X}$. By Lemma 3.4, $F^\star(H)$ has a Sylow tower of supersoluble type; therefore, $F^\star(H) = F(H)$.

(2) Since $P$ char $F^\star(H) = F(H)$ char $H$, the subgroup $P$ is normal in $G$. By Lemma 2.1(4), the subgroup $P_i$ is normal in $G$ for each $i$. Since $P$ is an arbitrary Sylow subgroup of $F^\star(H)$, every maximal subgroup of every Sylow subgroup of $F^\star(H)$ is normal in $G$.

(3) Since normal subgroups are $c$-normal, the group $G$ is supersoluble by Lemma 5.1. The theorem is proved.

**Corollary 5.1.** *Suppose that $H$ is a normal subgroup of a soluble group $G$ and the quotient group $G/H$ is supersoluble. If each maximal subgroup of every Sylow subgroup of $F(H)$ is normally embedded in $G$ then $G$ is supersoluble. In particular, if $G$ is a soluble group and each maximal subgroup of every Sylow subgroup of $F(G)$ is normally embedded in $G$ then the group $G$ is supersoluble.*

Proof. Since $H$ is soluble, $F^\star(H) = F(H)$. Applying Theorem 5.1, we infer that $G$ is supersoluble. For $G = H$ we obtain the second claim of the corollary.

Example 5.2. Let $G = H = \mathrm{SL}(2,5)$. Then $|F(H)| = 2$ and $F(H)$ satisfies all hypotheses of Corollary 5.1. But $G$ is not supersoluble. Therefore, the solubility of $G$ in Corollary 5.1 cannot be omitted.

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