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ON THE SOLVABILITY OF A FINITE GROUP WITH S-SEMINORMAL SCHMIDT SUBGROUPS

V. N. Knyagina, V. S. Monakhov, and E. V. Zubei UDC 512.542

A finite nonnilpotent group is called a Schmidt group if all its proper subgroups are nilpotent. A subgroup A is called S-seminormal (or SS-permutable) in a finite group G if there is a subgroup B such that $G = AB$ and A is permutable with every Sylow subgroup of B. We establish criteria for the solvability and π-solvability of finite groups in which some Schmidt subgroups are S-seminormal. In particular, we prove the solvability of a finite group in which all supersoluble Schmidt subgroups of even order are S-seminormal.

1. Introduction

We consider only finite groups. A nonnilpotent group in which all proper subgroups are nilpotent is called a Schmidt group. For the first time, these groups were considered by Schmidt in [1] who proved their biprimarity, normality of one Sylow subgroup, and cyclicity of the other subgroup. Surveys of the structure of Schmidt groups and their applications in the theory of finite groups can be found in [2, 3].

Since Schmidt groups are subgroups of every nonnilpotent group, they are universal subgroups of finite groups. For this reason, the properties of Schmidt subgroups contained in a group strongly affect the structure of the group itself. Groups with restrictions imposed on Schmidt subgroups were investigated in numerous works. Thus, the groups with subnormal Schmidt subgroups were studied in [4–6] and the groups with Hall Schmidt subgroups were investigated in [7].

A subgroup A is called *seminormal* in the group G if there exists a subgroup B such that $G = AB$ and AB_1 is a proper subgroup in G for every proper subgroup B_1 of B. It is clear that a subgroup of prime index is seminormal. A quasinormal subgroup (i.e., a subgroup permutable with all subgroups of a group) is seminormal. In the simple group $SL(2,4)$, the subgroup A isomorphic to the alternating group A_4 of degree 4 is a seminormal Schmidt subgroup but A is not quasinormal and not subnormal.

Some properties of seminormal subgroups were obtained in [8–11]. The criteria for the solvability of a group with some seminormal Schmidt subgroups were established in [12]. The results of the cited work are summarized in the following theorem:

Theorem 1.1.

1. *If A is a seminormal Schmidt subgroup of the group G and A^G is unsolvable, then $A/\Phi(A) \simeq A_4$ [12] (Theorem 1).*

2. *If all Schmidt $\{2,3\}$-subgroups of the group G are seminormal, then G is 3-solvable [12] (Theorem 2).*

3. *If all Schmidt $\{2,3\}$-subgroups and all 5-closed Schmidt $\{2,5\}$-subgroups of the group G are seminormal, then G is solvable [12] (corollary).*

Skorina Gomel' University, Belarus.

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There are different generalizations of the notion of seminormal subgroups (see [13]). In particular, the following notion is more general than quasinormality and seminormality:

A subgroup A is called S-*seminormal* (or SS-*permutable*) in the group G if there exists a subgroup B such that $G = AB$ and A is permutable with each Sylow subgroup from B. In this case, the subgroup B is called the S-*supplement* to A in G.

In any group, every subgroup whose index is equal to the power of a certain prime number p is S-seminormal and the Sylow p-subgroup of the group G is its S-supplement. The seminormal subgroups and S-quasinormal subgroups (i.e., subgroups permutable with all Sylow subgroups of the group [14]) are also S-seminormal. In the symmetric group S_4 of degree 4, the subgroup S_3 is S-seminormal but not seminormal and not S-quasinormal. The groups with some S-seminormal subgroups were investigated in numerous works (see, e.g., [15–20]).

In the present paper, we study groups in which some Schmidt subgroups are S-seminormal and establish criteria for the solvability and π-solvability of these groups. We prove the following theorem:

Theorem 1.2.

1. *If all supersolvable Schmidt subgroups of even order of the group G are S-seminormal in G, then G is solvable.*

2. *If all nonsupersolvable Schmidt subgroups of even order of the group G are S-seminormal, then the non-Abelian composition factors of the group G are isomorphic to $SL(2,4)$ or $SL(2,8)$. In particular, the group G is $\{2,3,5,7\}'$-solvable.*

3. *If all Schmidt $\{2,3\}$-subgroups of the group G are S-seminormal, then G is 3-solvable.*

4. *If all Schmidt $\{2,3\}$-subgroups and all 5-closed Schmidt $\{2,5\}$-subgroups of the group G are S-seminormal, then G is solvable.*

Note that Theorem 1.2 yields some new criteria of partial solvability for a group with seminormal Schmidt subgroups. These criteria supplement Theorem 1.1:

Corollary 1.1.

1. *If all supersolvable Schmidt subgroups of even order of the group G are seminormal in G, then G is solvable.*

2. *If all nonsupersolvable Schmidt subgroups of even order of the group G are seminormal, then the non-Abelian composition factors of the group G are isomorphic to $SL(2,4)$. In particular, the group G is $\{2,3,5\}'$-solvable.*

2. Definitions and Results

All notation and definitions used in the present work agree with [21, 22].

Let p be a prime number. A group with normal Sylow p-subgroup is called p-closed. A group containing a normal subgroup whose index coincides with the order of Sylow p-subgroup is called p-nilpotent. By $Z(G)$, $F(G)$, and $\Phi(G)$, we denote the center, the Fitting subgroup, and the Frattini subgroup of the group G, respectively. Also let H^G be the least normal subgroup in G containing the subgroup H. Symmetric and alternating groups of degree n are denoted by S_n and A_n, respectively, cyclic and elementary Abelian groups of degrees

m and p^t are denoted by Z_m and E_{p^t}, respectively, and $\pi(G)$ is the set of all prime divisors of the order of group G. If $|\pi(G)| = 1$, then the group G is called primary. For $|\pi(G)| = 2$, it is called biprimary. If $\pi \subseteq \pi(G)$, then $\pi' = \pi(G) \setminus \pi$.

A normal (subnormal) series of the group G is defined as the chain of subgroups

$$1 = G_0 \leq G_1 \leq \ldots \leq G_m = G$$

in which the subgroup G_i is normal in G (normal in G_{i+1}) for all $i = 0, 1, \ldots, m-1$. The quotient groups G_{i+1}/G_i are called quotients of this series. A normal series is called principal if G_{i+1}/G_i is the minimal normal subgroup of the group G/G_i for each i, and the numbers $|G_{i+1}/G_i|$, $i = 0, 1, \ldots, m-1$, are called indices of the principal series. A subnormal series is called composition if G_{i+1}/G_i is a simple group for each i.

A group G is called π-solvable, $\pi \subseteq \pi(G)$, if the indices of its principal series are either powers of prime numbers from π or are not divisible by prime numbers from π. A group G is called supersolvable if the indices of its principal series are prime numbers. The semidirect product of a subgroup A normal in G and a subgroup B is represented as follows: $G = [A]B$.

The lemma presented below establishes the properties of Schmidt groups obtained by Schmidt in 1924.

Lemma 2.1 [1]**.** *Suppose that S is a Schmidt group. Then the following assertions are true:*

(1) $S = [P]Q$, where P is a normal Sylow p-subgroup, Q is a nonnormal Sylow q-subgroup, p and q are different prime numbers;

(2) $Q = \langle y \rangle$ is a cyclic subgroup and $y^q \in Z(S)$;

(3) $|P/P'| = p^m$, where m is the exponent of number p modulo q;

(4) the principal series of the group S has the system of indices

$$p, p, \ldots, p, p^m, q, \ldots, q,$$

the number of indices equal to p coincides with n, where $p^n = |P'|$; the number of indices equal to q coincides with b, where $q^b = |Q|$.

In what follows, a Schmidt group with a normal Sylow p-subgroup P and a cyclic Sylow q-subgroup Q is called an $S_{\langle p,q \rangle}$-group.

Lemma 2.2 ([23], Lemma 1)**.** *An $S_{\langle p,q \rangle}$-group is supersolvable if and only if $|P| = p$ and q divides $p - 1$.*

Lemma 2.3.

1. *Every non-p-nilpotent group contains an $S_{\langle p,q \rangle}$-subgroup for some $q \in \pi(G)$ ([21], IV.5.4).*
2. *Every non-2-closed group contains an $S_{\langle q,2 \rangle}$-subgroup for some $q \in \pi(G)$ (see [24, p. 34], [25], 3.1.1).*

Example 2.1. For any odd p, an analog of Assertion 2 in Lemma 2.3 is not true. As counterexamples, we can mention the simple group $SL(2, 2^n)$ for $p = 3$ and any odd $n > 2$ or the group $PSL(2, p)$ for $p \geq 5$, respectively.

Lemma 2.4 ([12], Lemma 1)**.** *If K and D are subgroups of the group G, the subgroup D is normal in K, and K/D is an $S_{\langle p,q \rangle}$-subgroup, then the minimal supplement L to the subgroup D in K has the following* *properties:*

(1) L *is a* p*-closed* $\{p, q\}$*-subgroup;*

(2) *all proper normal subgroups in* L *are nilpotent;*

(3) L *contains an* $S_{\langle p,q\rangle}$*-subgroup* $[P]Q$ *such that* Q *is not contained in* D *and* $L = ([P]Q)^L = Q^L$*.*

Lemma 2.5 ([21], VI.4.10)**.** *Suppose that* A *and* B *are subgroups of the group* G *such that* $G \neq AB$ *and* $AB^g = B^g A$ *for all* $g \in G$*. Then either* $A^G \neq G$ *or* $B^G \neq G$*.*

Lemma 2.6. *Suppose that* A *is an* S*-seminormal subgroup of the group* G *and* B *is its* S*-supplement.*

1. *If* $A \leq H \leq G$, *then* A *is an* S*-seminormal subgroup of the group* H *and* $B \cap H$ *is an* S*-supplement to* A *in* H*.*
2. *If* N *is a normal subgroup of the group* G, *then* AN/N *is an* S*-seminormal subgroup in* G/N *and* BN/N *is an* S*-supplement to* AN/N *in* G/N*.*
3. *The subgroup* A *is permutable with* P^g *for all* $g \in G$ *and all Sylow subgroups* P *from* B*. In particular, the subgroup* B^g *is an* S*-supplement to the subgroup* A *for each* $g \in G$*.*

Proof. 1. By the Dedekind identity, $H = A(H \cap B)$. Let P be a Sylow subgroup from $H \cap B$ and let P_1 be a Sylow subgroup from B such that $P \leq P_1$. Then

$$P = P_1 \cap H, \qquad AP_1 = P_1 A,$$

$$AP = A(P_1 \cap H) = AP_1 \cap H = P_1 A \cap H = (P_1 \cap H)A = PA.$$

Hence, A is an S-seminormal subgroup in H and $B \cap H$ is an S-supplement to A in H.

2. Since $G = AB$, we get

$$G/N = (AN/N)(BN/N).$$

Let K/N be a Sylow p-subgroup in BN/N. Then there exists a Sylow subgroup P in BN such that

$$PN/N = K/N.$$

By Lemma VI.4.6 in [21], there exist Sylow p-subgroups P_1 in B and P_2 in N such that $P = P_1P_2$. It is clear that $PN = P_1N$. By the condition, the subgroup A is permutable with P_1. Hence, $A(PN) = (PN)A$ and

$$(AN/N)(PN/N) = (AN/N)(K/N) = (PN/N)(AN/N) = (K/N)(AN/N).$$

3. Let $g = ba,\ a \in A,\ b \in B$. Then $P^b \leq B$ and

$$AP^g = AP^{ba} = (AP^b)^a = (P^b A)^a = P^{ba} A = P^g A.$$

By virtue of $G = AB^g$, the subgroup B^g is an S-supplement to A in G.

Lemma 2.7. *If A is an S-seminormal subgroup of the group G and B is its S-supplement, then, for any element $g \in G$, the subgroup A^g is S-seminormal in G and the subgroup B is an S-supplement to the subgroup A^g.*

Proof. By the definition of S-seminormal subgroup and S-supplement, the group $G = AB$ and AP is a subgroup of the group G for any Sylow subgroup P from B. By Assertion 3 of Lemma 2.6, the subgroup A is permutable with $P^{g^{-1}}$ for any $g \in G$, i.e.,

$$AP^{g^{-1}} = P^{g^{-1}}A, \qquad A^gP = PA^g.$$

It follows from the equality $G = A^gB$ that A^g is an S-seminormal subgroup in the group G and B is an S-supplement to A^g in the group G.

Lemma 2.8. *If A is an S-seminormal subgroup of the group G and X is a nonempty set of elements from G, then the subgroup $A^X = \langle A^x \mid x \in X\rangle$ is S-seminormal in G and B is an S-supplement to A^X in G.*

Proof. Let B be an S-supplement to the subgroup A. By Lemma 2.7, the subgroup A^x, $x \in X$, is S-seminormal in G and the subgroup B is an S-supplement to A^x. By definition, each Sylow subgroup P from B is permutable with A^x. By Lemma 4 in [12], the subgroup P is permutable with A^X. It follows from $G = AB$ and $A \le A^X$ that

$$G = A^XB.$$

Thus, A^X is an S-seminormal subgroup in G and B is an S-supplement to A^X in G.

Lemma 2.9. *Suppose that all $S_{\langle p,q\rangle}$-subgroups in the group G are S-seminormal. The following assertions are true:*

(1) if H is a subgroup of the group G, then all $S_{\langle p,q\rangle}$-subgroups from H are S-seminormal in H;

(2) if N is a normal subgroup of the group G, then all $S_{\langle p,q\rangle}$-subgroups in the quotient group G/N are S-seminormal;

(3) if $N \le H \le G$ and N is normal in H, then all $S_{\langle p,q\rangle}$-subgroups in H/N are S-seminormal.

Proof. 1. The required assertion follows from Lemma 2.6 (1).

2. Let K/N be an $S_{\langle p,q\rangle}$-subgroup of the group G/N and let L be a minimal subgroup from K such that $K = LN$. By Lemma 2.4 (3), the subgroup L contains an $S_{\langle p,q\rangle}$-subgroup A for which $L = A^L$. By the condition, the subgroup A is S-seminormal in G and, by Lemma 2.8, the subgroup L is S-seminormal in G. Thus, according to Lemma 2.6 (2), the subgroup $LN/N = K/N$ is S-seminormal in G/N.

3. The required assertion follows from Assertions 1 and 2.

Lemma 2.10. *Suppose that A is a nontrivial S-seminormal subgroup of a simple group G. Then there exists a p-subgroup P, $p \in \pi(G)$, such that $G = AP$.*

Proof. Let B be an S-supplement to A in G and let P be a Sylow p-subgroup from B such that P is not contained in A. The subgroup P exists because $G = AB = A$, which contradicts $A \ne G$. Assume that $G \ne AP$. Then $G \ne AP^g$ for all $g \in G$. Since G is a simple group, $P^G = G$. By Lemma 2.6 (3), the subgroup A is permutable with P^g for each $g \in G$ and, hence, by Lemma 2.5, $A^G \ne G$, which contradicts the simplicity of the group G. Therefore, the assumption that $G \ne AP$ is not true and, hence, $G = AP$.

Proposition 2.1. *If, in a simple group G, there exists an S-seminormal Schmidt subgroup A, then one of the following assertions is true:*

(1) $G \simeq A_5,\ A \simeq [A_4]Z_3,\ B \simeq Z_5$;

(2) $G \simeq PSL(2,7),\ A \simeq [Z_7]Z_3,\ B \simeq D_8$;

(3) $G \simeq SL(2,8),\ A \simeq [E_8]Z_7,\ B \simeq Z_9$.

Here, B is an S-supplement to A in the group G.

Proof. The unsolvable groups $G = AB$, where A is a Schmidt group and B is a nilpotent group, are listed in Theorem 3 from [26]. For this factorization, $G/S(G)$ is isomorphic to one of the following groups: $PSL(2,7)$, $PGL(2,7)$, $SL(2,2^n)$, where $2^n - 1$ is a prime number, or $P\Gamma L(2,2^n)$ for a certain prime number n. In the analyzed case, the group G is simple and $|\pi(G)| = 3$ by Lemma 2.10. Hence, for the group $SL(2,2^n)$, the only possible case is $n \in \{2,3\}$. Thus,

$$G \in \bigl\{PSL(2,7), PSL(2,5) = SL(2,4) = A_5, SL(2,8)\bigr\}.$$

The factorizations of these groups are known. The required factorizations are indicated in Assertions 1–3 of Proposition 2.1.

3. Proof of Theorems 1.2

Proof of Assertion 1. Let all supersolvable Schmidt subgroups of even order in the group G be S-seminormal. Assume that the group G is unsolvable and H/K is a non-Abelian composition factor of the group G. Then H/K is a simple group of even order and this group is not 2-closed. By Lemma 2.3 (2), in H/K, there exists an $S_{\langle p,2\rangle}$-subgroup A/K for some $p \in \pi(G)$. By Lemma 2.2, each $S_{\langle p,2\rangle}$-subgroup from G is supersolvable and, by condition, S-seminormal. By Lemma 2.9(3), the subgroup A/K is S-seminormal in H/K and, hence, it is possible to apply Proposition 2.1 (with the exception of the case where A/K is an $S_{\langle p,2\rangle}$-semigroup). Therefore, the assumption that G is an unsolvable group is not true and the group G is solvable. Assertion 1 of Theorem 1.2 is proved.

Proof of Assertion 2. Let all nonsupersolvable Schmidt subgroups of even order in the group G be S-seminormal. Assume that the group G is unsolvable and H/K is a non-Abelian composition factor of the group G. Then H/K is a simple group of even order and, hence, it is not 2-nilpotent. By Lemma 2.3(1), in H/K, there exists an $S_{\langle 2,q\rangle}$-subgroup A/K for some $q \in \pi(G)$. By Lemma 2.2, every $S_{\langle 2,q\rangle}$-subgroup is nonsupersolvable. By Lemma 2.9(3) and the condition, the subgroup A/K is S-seminormal in H/K and it is possible to apply Proposition 2.1.

If $H/K \simeq SL(2,4)$, then $A/K \simeq A_4 = [E_4]Z_3$. In $SL(2,4)$, the Schmidt subgroups are exhausted (to within conjugacy) by the following subgroups: A_4, $[Z_5]Z_2$, and $[Z_3]Z_2$. The subgroups $[Z_5]Z_2$ and $[Z_3]Z_2$ are supersolvable and the subgroup A_4 is nonsupersolvable and seminormal. Hence, the group $SL(2,4)$ can be a composition factor of the group G.

If $H/K \simeq SL(2,8)$, then $A/K \simeq [E_8]Z_7$. In $SL(2,8)$, the Schmidt subgroups are exhausted (to within conjugacy) by the following subgroups: $[E_8]Z_7$, $[Z_7]Z_2$, and $[Z_3]Z_2$. Among these subgroups, only the subgroup $[E_8]Z_7$ is nonsupersolvable and it is S-seminormal in H/K. Hence, the group $SL(2,8)$ can be a composition factor of the group G.

An isomorphism of H/K with the group $PSL(2,7)$ is excluded and $PSL(2,7)$ is a nonsupersolvable Schmidt subgroup A_4, which is not S-seminormal in $PSL(2,7)$.

Thus, the non-Abelian composition factors of the group G are exhausted by the groups $SL(2,4)$ and $SL(2,8)$ of the orders $2^2 \cdot 3 \cdot 5$ and $2^3 \cdot 3^2 \cdot 7$, respectively. Therefore, the non-Abelian composition factors of the group G are $\{2,3,5,7\}$-groups and G is $\{2,3,5,7\}'$-solvable.

Assertion 2 of Theorem 1.2 is proved

Proof of Assertion 3. Assume that all Schmidt $\{2,3\}$-subgroups of the group G are S-seminormal. By induction on the order of the group $G,$ we prove that the group G is 3-solvable. Let N be a normal subgroup of the group G. By Lemma 2.9(1), (2), all Schmidt $\{2,3\}$-subgroups in the subgroup N and in the quotient group G/N are seminormal. If $1 \neq N \neq G,$ then, by induction, the subgroup N and the quotient group G/N are 3-solvable. This implies that the group G is 3-solvable. Therefore, the group G can be regarded as simple.

Assume that the group G contains a Schmidt $\{2,3\}$-subgroup A. By Proposition 2.1, $A \simeq A_4$ and $G \simeq SL(2,4)$. However, the group $SL(2,4)$ contains the Schmidt subgroup $[Z_3]Z_2,$ which is not S-seminormal. We arrive at a contraction.

Hence, the group G does not contain Schmidt $\{2,3\}$-subgroups. We now check that, in this case, the group G is S_4-free. Assume the contrary, i.e., that there exist subgroups U and V such that U is normal in V and $V/U \simeq S_4$. The group S_4 contains a subgroup $S_3,$ which is an $S_{\langle 3,2\rangle}$-subgroup. By Lemma 2.4, the subgroup V contains an $S_{\langle 3,2\rangle}$-subgroup. We arrive at a contraction. Hence, the group G is S_4-free. By Theorem 4.174 in [27], either a Sylow 2-subgroup in G is Abelian or $G \in \{Sz(2^n), U(3,2^n)\},$ n is odd. Simple groups with Abelian Sylow 2-subgroup are known (see Theorem 4.126 in [27]). Each of these groups contains a non-Abelian subgroup of order 6, i.e., is an $S_{\langle 3,2\rangle}$-subgroup. The subgroup $U(3,2^n)$, where n is odd, also contains an $S_{\langle 3,2\rangle}$ subgroup [27] (Theorem 4.168). Hence, these groups are excluded. Since the order of the Suzuki group $Sz(2^n)$ is not divisible by 3, this group is 3-solvable.

Assertion 3 of Theorem 1.2 is proved.

Proof of Assertion 4. Assume that all Schmidt $\{2,3\}$-subgroups and all 5-closed Schmidt $\{2,5\}$-subgroups of the group G are S-seminormal. By induction on the order of the group, we prove that the group G is solvable. As in the proof of Assertion 3, we show that G is a simple group. By virtue of Assertion 3, the group G is 3-solvable. Hence, G is a simple $3'$-group. By the Thompson theorem, $G \simeq Sz(2^n)$. By Theorems XI.3.6 and XI.3.10 in [28], the group G contains an $S_{<5,2>}$-subgroup A. Since A is an S-seminormal subgroup in G, we can use Proposition 2.1. Thus, the order of the group G is divisible by 3. We arrive at a contraction.

Assertion 4 of Theorem 1.2 is proved.

Since the seminormal subgroups are S-seminormal, Assertion 1 in Theorem 1.2 yields Assertion 1 in Corollary 1.1.

Proof of Assertion 2 in Corollary 1.1. Assume that all nonsupersolvable Schmidt subgroups of even order in the group G are seminormal. Since each seminormal subgroup is S-seminormal, we can use Assertion 2 of Theorem 1.2 . The Schmidt subgroup $[E_8]Z_7$ of the group $SL(2,8)$ is S-seminormal, its index is equal to 9, and it is not seminormal. Hence, $SL(2,8)$ does not contain seminormal Schmidt subgroups of even order and this group is excluded. Thus, the non-Abelian composition factors of the group G are isomorphic to $SL(2,4)$.

4. Criteria for the p-Solvability of a Group with S-Seminormal Schmidt Subgroups of Odd Order

A group whose order is divisible by a prime number p is called a pd-group.

The assertions similar to Assertions 1–3 of Theorem 1.2 for groups with S-seminormal Schmidt subgroups of odd order are not true. In the simple groups $SL(2,2^n),$ $n \geq 2,$ and $Sz(2^{2m+1}),$ $m \geq 1,$ all subgroups of odd order are nilpotent. Hence, these groups do not contain Schmidt subgroups of odd orders. Therefore, the indicated groups (and not only these groups) can play the role of composition factors for the groups with S-seminormal Schmidt subgroups of odd order.

In addition, by Lemma 2.2, a Schmidt group of even order is either supersolvable (if it is 2-nilpotent) or nonsupersolvable (if it is 2-closed). The property of alternation does not hold for Schmidt groups of odd order. Thus, any Schmidt $\{3, 5\}$-group (both 3-closed and 3-nilpotent) is nonsupersolvable. Hence, for odd p, we can separate the set of Schmidt pd-subgroups into p-closed and p-nilpotent.

Theorem 4.1. *Suppose that p is an odd prime number and all p-closed Schmidt pd-subgroups of the group G are S-seminormal. If G is not p-solvable, then $p = 7$ and a non-p-solvable composition factor of the group G is isomorphic to $PSL(2, 7)$.*

Proof. Assume that the group G is not p-solvable and H/K is not a p-solvable composition factor of the group G. Then H/K is a simple pd-group. Hence, it is not p-nilpotent. By Lemma 2.3(1), in H/K, there exists an $S_{\langle p,q\rangle}$-subgroup A/K for some $q \in \pi(G)$. By the condition, all $S_{\langle p,q\rangle}$-subgroups of the group G are S-seminormal in G. By Lemma 2.9(3), the subgroup A/K is S-seminormal in H/K. Hence, it is possible to apply Proposition 2.1. Since $p > 2$, we get $H/K \simeq PSL(2, 7)$ and $p = 7$.

The theorem is proved.

Corollary 4.1. *If p is an odd prime number and all p-closed Schmidt pd-subgroups of the group G are seminormal, then G is p-solvable.*

Proof. Assume that the group G is non-p-solvable and that H/K is a non-p-solvable composition factor of the group G. By Theorem 4.1, $H/K \simeq PSL(2, 7)$ and $p = 7$. In the group $PSL(2, 7)$, the 7-closed Schmidt $7d$-subgroup $A = [Z_7]Z_3$ has index 8 and the subgroup A is S-seminormal but not seminormal. Therefore, the group $PSL(2, 7)$ is excluded. Thus, the assumption is not true and the group G is p-solvable.

Example 4.1. As indicated in Example 2.1, $SL(2, 2^n)$ does not contain 3-nilpotent Schmidt $3d$-subgroups for any odd n, while $PSL(2, p)$ does not contain p-nilpotent Schmidt pd-subgroups for any $p \geq 5$. Therefore, the indicated groups (and not only these groups) can play the role of composition factors for a group with S-seminormal p-nilpotent Schmidt pd-subgroups.

Thus, the description of composition factors of an unsolvable group with nonsupersolvable (supersolvable) S-seminormal Schmidt subgroups of odd order depends on the solution of the following problem:

To list all simple groups with supersolvable (nilpotent) subgroups of odd orders.

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