J. Group Theory **21** (2018), 273–287
DOI 10.1515/jgth-2017-0039 

# On groups with formational subnormal Sylow subgroups

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Communicated by Robert M. Guralnick

**Abstract.** We investigate a finite group $G$ with $\mathfrak{F}$-subnormal Sylow subgroups, where $\mathfrak{F}$ is a subgroup-closed formation and $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. We prove that $G$ is soluble and the derived subgroup of each metanilpotent subgroup is nilpotent. We also describe the structure of groups in which every Sylow subgroup is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal.

## 1 Introduction

All groups in this paper are finite. We use the standard notation and terminology of [3,10]. The formations of all abelian, nilpotent, supersoluble and soluble groups are denoted by $\mathfrak{A}$, $\mathfrak{N}$, $\mathfrak{U}$ and $\mathfrak{S}$, respectively. We also use the following notation:

- $\mathfrak{E}$ denotes the formation of all finite groups,
- $\mathfrak{A}_1$ denotes the formation of all abelian groups with elementary abelian Sylow subgroups,
- $\mathcal{A}$ denotes the formation of all soluble groups with abelian Sylow subgroups.

Let $\mathfrak{F}$ be a formation, and let $G$ be a group. The subgroup

$$G^{\mathfrak{F}} = \bigcap\{N \lhd G : G/N \in \mathfrak{F}\}$$

is called the $\mathfrak{F}$-residual of $G$. If $\mathfrak{X}$ and $\mathfrak{F}$ are subgroup-closed formations, then the product $\mathfrak{X}\mathfrak{F} = \{G \in \mathfrak{E} : G^{\mathfrak{F}} \in \mathfrak{X}\}$ is also a subgroup-closed formation by [10, Theorem 5.10 (3)] and [3, Definition IV.1.7].

The $\mathfrak{F}$-subnormality and $\mathfrak{F}$-abnormality could be regarded as the extension of the subnormality and abnormality to formation theory, see [3, Definitions IV.5.12, Remarks IV.5.6] and [1, Section 6.1].

A subgroup $H$ is called an $\mathfrak{F}$-subnormal subgroup of a group $G$ if there is a chain of subgroups

$$H = H_0 \lessdot H_1 \lessdot \cdots \lessdot H_n = G \tag{1.1}$$

such that $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all $i$. This is equivalent to $H_i^{\mathfrak{F}} \leq H_{i-1}$. Here $Y_X = \bigcap_{x\in X} Y^x$ is the core of a subgroup $Y$ in a group $X$, $H_{i-1} \lessdot H_i$ denotes that $H_{i-1}$ is a maximal subgroup of a group $H_i$.

A subgroup $H$ of a group $G$ is said to be $\mathfrak{F}$-abnormal in $G$ if $L/K_L \notin \mathfrak{F}$ for all subgroups $K$ and $L$ such that $H \leq K \lessdot L \leq G$.

In any group $G$, there are no proper subgroups that are both $\mathfrak{F}$-subnormal and $\mathfrak{F}$-abnormal. It is clear that for formations $\mathfrak{F}$ and $\mathfrak{X}$, $\mathfrak{F} \subseteq \mathfrak{X}$, every $\mathfrak{F}$-subnormal subgroup is $\mathfrak{X}$-subnormal and every $\mathfrak{X}$-abnormal subgroup is $\mathfrak{F}$-abnormal.

Groups with certain $\mathfrak{F}$-subnormal subgroups were investigated in [4, 11–16, 18–20].

T. I. Vasil'eva and A. F. Vasil'ev [18] proposed to denote the class of all groups in which every Sylow subgroup is $\mathfrak{F}$-subnormal by w$\mathfrak{F}$. In any soluble group, every Sylow subgroup is $\mathfrak{A}_1\mathfrak{N}$-subnormal (see Corollary 3.7). Therefore in the soluble universe, the class w$\mathfrak{F}$ should be investigated when $\mathfrak{A}_1\mathfrak{N} \not\subseteq \mathfrak{F}$. Since $\mathfrak{N}$-subnormal subgroups are subnormal [17, Section II.8], we have w$\mathfrak{N} = \mathfrak{N}$. The detailed description of the class w$\mathfrak{U}$ and properties of groups from this class are obtained in [11, 12, 19].

In this paper, we investigate the class w$\mathfrak{F}$ when $\mathfrak{F}$ is a subgroup-closed formation and $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. We get the following characterizations of this class.

**Theorem A.** *Let $\mathfrak{F}$ be a subgroup-closed formation and let $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. The following statements hold.*

(1) *Every Sylow subgroup of a group $G$ is $\mathfrak{F}$-subnormal if and only if $G^{\mathcal{A}}$ is nilpotent.*

(2) *Every Sylow subgroup of a group $G$ is $\mathfrak{F}$-subnormal if and only if $G$ is soluble and every its metanilpotent subgroup has the nilpotent derived subgroup.*

Note that statement (1) of Theorem A is equivalent to w$\mathfrak{F} = \mathfrak{N}\mathcal{A}$.

In Section 4, we use Theorem A to investigate a group in which every Sylow subgroup is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal. We prove

**Theorem B.** *Let $\mathfrak{F}$ be a subgroup-closed formation and let $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. Every Sylow subgroup of a group $G$ is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal of nilpotency class at most 2 if and only if either $G \in \mathfrak{N}\mathcal{A}$ or $G = G^{\mathfrak{N}} \rtimes P$, where $P$ is a non-abelian $\mathfrak{F}$-abnormal Sylow $p$-subgroup of $G$ for some $p \in \pi(G)$ and the Carter and Gaschütz subgroup of $G$, $P' \leq Z(P)$, $G^{\mathfrak{N}} = G^{\mathfrak{U}} \in \mathfrak{N}\mathcal{A}$.*

## 2 Preliminaries

We write $X \leq Y$ and $X \trianglelefteq Y$ if $X$ is a subgroup of a group $Y$ and $X$ is a normal subgroup of $Y$, respectively. If $X \neq Y$, then we use $X < Y$ and $X \triangleleft Y$. The semidirect product of a subgroup $A$ and a normal subgroup $B$ is denoted by $A \rtimes B$.

We use $Z(G)$, $\Phi(G)$ and $F(G)$ to denote the centre, Frattini and Fitting subgroups of a group $G$, respectively. The derived subgroup of a group $G$ is denoted by $G'$.

A nilpotent group $P$ has nilpotency class at most 2 if $P' \leq Z(P)$.

**Lemma 2.1.** *Let $\mathfrak{F}$ be a formation, let $H$ and $K$ be subgroups of a group $G$ and let $N \trianglelefteq G$. The following statements hold.*

(1) *If $K$ is $\mathfrak{F}$-subnormal in $H$ and $H$ is $\mathfrak{F}$-subnormal in $G$, then $K$ is $\mathfrak{F}$-subnormal in $G$,* [1, Lemma 6.1.6 (1)].

(2) *If $K/N$ is $\mathfrak{F}$-subnormal in $G/N$, then $K$ is $\mathfrak{F}$-subnormal in $G$,* [1, Lemma 6.1.6 (2)].

(3) *If $H$ is $\mathfrak{F}$-subnormal in $G$, then $HN/N$ is $\mathfrak{F}$-subnormal in $G/N$,* [1, Lemma 6.1.6 (3)].

(4) *If $\mathfrak{F}$ is a subgroup-closed formation and $G^{\mathfrak{F}} \leq K$, then $K$ is $\mathfrak{F}$-subnormal in $G$,* [1, Lemma 6.1.7 (1)].

(5) *If $\mathfrak{F}$ is a subgroup-closed formation and $H$ is $\mathfrak{F}$-subnormal in $G$, then $H \cap K$ is $\mathfrak{F}$-subnormal in $K$,* [1, Lemma 6.1.7 (2)].

(6) *If $\mathfrak{F}$ is a subgroup-closed formation and $H \leq K \leq G \in \mathfrak{F}$, then $H$ is $\mathfrak{F}$-subnormal in $K$,* [1, Lemma 6.1.7 (1)].

Throughout this paper $\mathbb{P}$ denotes the set of all primes.

**Lemma 2.2.** *Let $\mathfrak{F}$ be a formation containing a group of order $p$ for all primes $p$, and let $A$ be an $\mathfrak{F}$-abnormal subgroup of a group $G$. The following statements hold.*

(1) *If $A \leq B \leq G$, then $A$ is $\mathfrak{F}$-abnormal in $B$ and $A = N_G(A)$.*

(2) *If $A \leq B \leq G$, then $B$ is $\mathfrak{F}$-abnormal in $G$ and $B = N_G(B)$.*

*Proof.* (1) It is clear that $A$ is $\mathfrak{F}$-abnormal in $B$. Assume that there is a subgroup $K$ of $G$ such that $A \leq K$ and $K \neq N_G(K)$. Hence there is a subgroup $L$ such that $K < L \leq N_G(K)$, $|L/K| \in \mathbb{P}$. By hypothesis, $L/K \in \mathfrak{F}$. This contradicts the $\mathfrak{F}$-abnormality of $A$. Therefore $K = N_G(K)$ for every subgroup $K$ containing $A$, in particular, $A = N_G(A)$.

(2) Let $A \leq B \leq G$. By definition, $B$ $\mathfrak{F}$-abnormal in $G$. As in (1) we get $B = N_G(B)$. □

**Lemma 2.3** ([8, Lemma 1]). *The following statements hold.*

(1) *If $K \leq H \leq G$, then $K_G \leq K_H$.*

(2) *If $N \leq H \leq G$ and $N \lhd G$, then $N \leq H_G$ and $(H/N)_{G/N} = H_G/N$.*

(3) *If $N \lhd G$ and $H \leq G$, then $(H_G)N \leq (HN)_G$.*

**Lemma 2.4.** *Let $\mathfrak{F}$ be a formation, let $H \leq G$ and $N \trianglelefteq G$. The following statements hold.*

(1) *If $H$ is $\mathfrak{F}$-abnormal in $G$, then $HN/N$ is $\mathfrak{F}$-abnormal in $G/N$.*

(2) *if $N \leq H$ and $H/N$ is $\mathfrak{F}$-abnormal in $G/N$, then $H$ $\mathfrak{F}$-abnormal in $G$.*

*Proof.* (1) Let

$$HN/N \leq K/N \lessdot L/N \leq G/N. \tag{2.1}$$

It follows that $H \leq HN \leq K \lessdot L \leq G$. Since $H$ is $\mathfrak{F}$-abnormal in $G$, we have $L/K_L \notin \mathfrak{F}$. By Lemma 2.3 (2),

$$(L/N)/(K/N)_{L/N} = (L/N)/(K_L/N) \simeq L/K_L \notin \mathfrak{F}. \tag{2.2}$$

Hence $HN/N$ is $\mathfrak{F}$-abnormal in $G/N$.

(2) Assume that $N \leq H \leq K \lessdot L \leq G$. Since $H/N$ is $\mathfrak{F}$-abnormal in $G/N$, in view of Lemma 2.3 (2), we get (2.1) and (2.2). Hence $H$ is $\mathfrak{F}$-abnormal in $G$. □

Let $\mathfrak{F}$ be a class of groups. A group $G$ is called a minimal non-$\mathfrak{F}$-group if $G \notin \mathfrak{F}$ but every proper subgroup of $G$ belongs to $\mathfrak{F}$.

**Lemma 2.5.** *Let $\mathfrak{F}$ be a formation. If $G$ is a minimal non-$\mathfrak{F}$-group, $N \triangleleft G$ and $G/N \notin \mathfrak{F}$, then $N \leq \Phi(G)$.*

*Proof.* Suppose that $N \not\leq \Phi(G)$. Then in $G$ there is a maximal subgroup $M$ such that $G = MN$. Since $G$ is a minimal non-$\mathfrak{F}$-group, it follows that $M \in \mathfrak{F}$ and $G/N \simeq M/(M \cap N) \in \mathfrak{F}$, a contradiction. Thus, $N \leq \Phi(G)$. □

**Lemma 2.6.** *Suppose that $\mathfrak{F}$ is a (subgroup-closed) formation. Then $\mathfrak{N}\mathfrak{F}$ is a saturated (subgroup-closed) formation.*

*Proof.* The product of (subgroup-closed) formations is a (subgroup-closed) formation [10, Theorem 5.10 (2)], hence $\mathfrak{N}\mathfrak{F}$ is a (subgroup-closed) formation. Let $G/\Phi(G) \in \mathfrak{N}\mathfrak{F}$. Then in $G/\Phi(G)$ there is a nilpotent normal subgroup $K/\Phi(G)$ such that

$$G/K \simeq (G/\Phi(G))/(K/\Phi(G)) \in \mathfrak{F}, \quad \Phi(G) \leq K \triangleleft G, \quad K/\Phi(G) \in \mathfrak{N}.$$

In view of [10, Theorem 3.24], $K$ is nilpotent and $G \in \mathfrak{N}\mathfrak{F}$. □

**Lemma 2.7.** *A minimal non-$\mathfrak{A}$-group is primary non-abelian group in which all proper subgroups are abelian. Conversely, every primary non-abelian group with abelian proper subgroups is a minimal non-$\mathfrak{A}$-group.*

*Proof.* Assume that $G$ is a minimal non-$\mathcal{A}$-group and $P$ is a Sylow subgroup of $G$. If $G \neq P$, then $P$ is abelian and $G$ is $\mathcal{A}$-group, a contradiction. Hence $G = P$ is a primary non-abelian group. If $P_1 < P$, then $P_1$ coincides with its Sylow subgroup. Therefore $P_1$ is abelian. Thus a minimal non-$\mathcal{A}$-group is a non-abelian primary group in which all proper subgroups are abelian. The converse is obvious. □

**Lemma 2.8.** *Let $G$ be a soluble minimal non-$\mathfrak{N}\mathcal{A}$-group. The following statements hold.*

(1) $G = P \rtimes Q$.

(2) $P = G^{\mathfrak{N}\mathcal{A}}$ *is a Sylow $p$-subgroup, its properties are described in* [17, Theorem 24.2]*; in particular, $P/\Phi(P)$ is a minimal normal subgroup in $G/\Phi(G)$.*

(3) *$Q$ is a non-abelian Sylow $q$-subgroup in which all proper subgroups are abelian.*

(4) $Q' \leq C_G(\Phi(P))$.

*Proof.* By Lemma 2.6, $\mathfrak{N}\mathcal{A}$ is a saturated subgroup-closed formation. In view of [2, Proposition 1], $G/G^{\mathfrak{N}\mathcal{A}}$ is a minimal non-$\mathcal{A}$-group. By Lemma 2.7, $G/G^{\mathfrak{N}\mathcal{A}}$ is a primary non-abelian group in which all proper subgroup are abelian. Hence $G^{\mathfrak{N}\mathcal{A}}$ is a Sylow subgroup of $G$ and $G = P \rtimes Q$, where $P = G^{\mathfrak{N}\mathcal{A}}$. The properties of $P$ are described in [17, Theorem 24.2]. In particular, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(G)$. It follows that $H = \Phi(P) \rtimes Q$ is a maximal subgroup of $G$. By hypothesis, $H \in \mathfrak{N}\mathcal{A}$, so $H/F(H) \in \mathcal{A}$. Since $\Phi(P) \subseteq F(H)$, $H/F(H)$ is an abelian $q$-group and $Q' \leq F(H)$, i.e. $Q' \leq C_G(\Phi(P))$. □

Let $G$ be a group and let $\mathfrak{X}$ be a class of groups. A subgroup $H$ of a group $G$ is $\mathfrak{X}$-maximal in $G$ if $H \in \mathfrak{X}$ and $H = K$ whenever $H \leq K \leq G$ and $K \in \mathfrak{X}$. A subgroup $H$ is an $\mathfrak{X}$-projector of $G$ if $HN/N$ is an $\mathfrak{X}$-maximal subgroup of $G/N$ for any normal subgroup $N$ of $G$.

**Lemma 2.9** ([17, Theorem 15.1]). *Let $\mathfrak{F}$ be a formation. A subgroup $H$ of a soluble group $G$ is an $\mathfrak{F}$-projector of $G$ if and only if $H \in \mathfrak{F}$ and $H$ is $\mathfrak{F}$-abnormal in $G$.*

If $G$ has a maximal subgroup $M$ with trivial core, then $G$ is said to be primitive and $M$ is its primitivator [5].

**Lemma 2.10** ([9, Lemma 8]). *Let $\mathfrak{F}$ be a saturated formation and let $G$ be a group. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all nontrivial normal subgroups $N$ of $G$. Then $G$ is a primitive group.*

**Lemma 2.11** ([10, Theorems 4.41 and 4.42]). *Let $G$ be a soluble primitive group with a primitivator $M$. The following statements hold.*

(1) $\Phi(G) = 1$.

(2) $F(G) = C_G(F(G)) = O_p(G)$ *for some* $p \in \pi(G)$.

(3) *$G$ has a unique minimal normal subgroup $N \in \mathfrak{A}_1$, furthermore $N = F(G)$.*

(4) *$G = N \rtimes M$ and $O_p(M) = 1$.*

**Lemma 2.12.** *In a soluble group, every subnormal subgroup is $\mathfrak{A}_1$-subnormal.*

*Proof.* Let $H$ be a subnormal subgroup of a soluble group $G$. There is a composition series of $G$ containing $H$. Since $G$ is soluble, the composition factors are of prime orders. Hence there is a chain of subgroups

$$H = H_0 \lessdot H_1 \lessdot \cdots \lessdot H_n = G$$

such that $H_i \lhd H_{i+1}$ and $|H_{i+1} : H_i| \in \mathbb{P}$. Thus, $H_{i+1}/H_i \in \mathfrak{A}_1$ for all $i$, and so $H$ is $\mathfrak{A}_1$-subnormal in $G$. □

**Lemma 2.13.** *A group $G$ is soluble if and only if $G$ contains an $\mathfrak{S}$-subnormal soluble subgroup.*

*Proof.* If $G$ is a soluble group, then every subgroup of $G$ is soluble and $\mathfrak{S}$-subnormal by Lemma 2.1 (6). Conversely, assume that $G$ contains an $\mathfrak{S}$-subnormal soluble subgroup $H$. Since $H$ is a proper subgroup of $G$ and $\mathfrak{S}$-subnormal in $G$, there is a maximal subgroup $M$ containing $H$ such that $G/M_G$ is soluble. By Lemma 2.1 (5), $H$ is $\mathfrak{S}$-subnormal in $M$ and, by induction, $M$ is soluble. Thus $G/M_G$ and $M_G$ are soluble, hence $G$ is soluble. □

## 3 Groups with $\mathfrak{F}$-subnormal Sylow subgroups

In this section, we investigate groups that belong to $\mathrm{w}\mathfrak{F}$ on condition that $\mathfrak{F}$ is a subgroup-closed formation and $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$.

**Example 3.1.** In the symmetric group $S_4$ of degree 4, every Sylow subgroup is $\mathfrak{A}_1\mathfrak{A}$-subnormal, i.e. $S_4 \in \mathrm{w}(\mathfrak{A}_1\mathfrak{A}) \subseteq \mathrm{w}\mathfrak{F}$ for any formation $\mathfrak{F}$ with $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F}$.

**Example 3.2.** The general linear group $\mathrm{GL}(2,3)$ of order $2^4 3$ has a subgroup chain

$$1 \le P \times Z \le \mathrm{SL}(2,3) = Q \rtimes P \lhd \mathrm{GL}(2,3), \quad 1 \le Q \lhd R \lessdot \mathrm{GL}(2,3),$$

where $Z = Z(\mathrm{GL}(2,3))$, $P$ is the Sylow 3-subgroup and $R$ is the Sylow 2-subgroup of $\mathrm{GL}(2,3)$, $Q$ is the quaternion group of order 8, $Q \lhd \mathrm{GL}(2,3)$. It follows that $\mathrm{GL}(2,3) \in \mathrm{w}(\mathfrak{A}_1\mathfrak{A}) \subseteq \mathrm{w}\mathfrak{F}$ for any formation $\mathfrak{F}$ such that $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F}$.

The following example shows that a subgroup-closed formation $\mathfrak{F}$ on condition that $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$ could be nonsaturated.

**Example 3.3.** Let $\mathfrak{F} = \mathrm{Sform}\{\mathfrak{A}_1\mathfrak{A} \cup S_4\}$ be a subgroup-closed formation generated by the formation $\mathfrak{A}_1\mathfrak{A}$ and the symmetric group $S_4$. Then $\mathfrak{F}$ is not saturated and $\mathfrak{A}_1\mathfrak{A} \subset \mathfrak{F} \subset \mathfrak{N}\mathcal{A}$, since $S_4 \in \mathfrak{F} \setminus \mathfrak{A}_1\mathfrak{A}$ and $\mathrm{GL}(2,3) \in \mathfrak{N}\mathcal{A} \setminus \mathfrak{F}$. We can similarly construct a subgroup-closed nonsaturated formation $\mathrm{Sform}\{\mathfrak{A}_1\mathfrak{A} \cup V\}$ for any group $V \in \mathfrak{N}\mathcal{A} \setminus \mathfrak{A}_1\mathfrak{A}$.

**Lemma 3.4.** *If $\mathfrak{F}$ is a subgroup-closed soluble formation, then $\mathrm{w}\mathfrak{F}$ is also a subgroup-closed soluble formation.*

*Proof.* By [18, Lemma 1.4], $\mathrm{w}\mathfrak{F}$ is subgroup-closed and, by Lemma 2.13, $\mathrm{w}\mathfrak{F}$ is soluble. □

**Proposition 3.5.** *Let $\mathfrak{F}$ be a subgroup-closed formation, let $G$ be a soluble group and let $H \leq G$.*

(1) *If $H \in \mathfrak{F}$, then $H$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $G$.*

(2) *$H$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $G$ if and only if $H$ is $\mathfrak{N}\mathfrak{F}$-subnormal.*

(3) *A subgroup $H$ is $\mathfrak{A}_1\mathfrak{F}$-abnormal in $G$ if and only if $H$ is $\mathfrak{N}\mathfrak{F}$-abnormal.*

*Proof.* (1) We use induction on $|G|$. Assume that $H \in \mathfrak{F}$ and $N$ is a minimal normal subgroup of $G$. By induction, $HN/N$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $G/N$. Hence $HN$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $G$ by Lemma 2.1 (2). Since $HN \in \mathfrak{A}_1\mathfrak{F}$, we conclude that $H$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $HN$, and so $H$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $G$ in view of Lemma 2.1 (1).

(2) Suppose that $H$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $G$. Since $\mathfrak{A}_1\mathfrak{F} \subseteq \mathfrak{N}\mathfrak{F}$, we deduce that $H$ is $\mathfrak{N}\mathfrak{F}$-subnormal in $G$. To prove the converse, we use induction on $|G|$. Let $H$ be an $\mathfrak{N}\mathfrak{F}$-subnormal subgroup of $G$, $M$ be a maximal subgroup of $G$ such that $H \leq M$ and $G/M_G \in \mathfrak{N}\mathfrak{F}$. Since $G/M_G$ is primitive, by Lemma 2.11,

$$G/M_G = \overline{G} = \overline{N} \rtimes \overline{M}, \quad \overline{N} = F(\overline{G}) = C_{\overline{G}}(F(\overline{G})),$$

$\overline{N}$ is a minimal normal subgroup of $\overline{G}$, $\overline{N} \in \mathfrak{A}_1$. As $G/M_G \in \mathfrak{N}\mathfrak{F}$ and $\overline{N} = F(\overline{G})$, we have $\overline{M} \in \mathfrak{F}$. Now, $G/M_G \in \mathfrak{A}_1\mathfrak{F}$ and $M$ is $\mathfrak{A}_1\mathfrak{F}$-normal in $G$. Since $H$ is $\mathfrak{N}\mathfrak{F}$-subnormal in $G$, by induction, $H$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $M$. Thus, $H$ is $\mathfrak{A}_1\mathfrak{F}$-subnormal in $G$ by Lemma 2.1 (1).

(3) Suppose that $H$ is $\mathfrak{A}_1\mathfrak{F}$-abnormal in $G$. Then $L/K_G \notin \mathfrak{A}_1\mathfrak{F}$ for any subgroups $K$ and $L$ such that $H \leq K \lessdot L \leq G$. Since $L/K_G$ is a primitive group, we obtain

$$L/K_G = N/K_G \rtimes M/K_G, \quad N/K_G = F(L/K_G) \in \mathfrak{A}_1,$$

in view of Lemma 2.11. If $L/K_G \in \mathfrak{N}\mathfrak{F}$, then $L/K_G \in \mathfrak{A}_1\mathfrak{F}$, a contradiction. So $L/K_G \notin \mathfrak{N}\mathfrak{F}$ and $H$ is $\mathfrak{N}\mathfrak{F}$-abnormal in $G$. Conversely, if $H$ $\mathfrak{N}\mathfrak{F}$-abnormal in $G$, then $L/K_G \notin \mathfrak{N}\mathfrak{F}$ for any subgroups $K$ and $L$ such that $H \leq K \lessdot L \leq G$. As $\mathfrak{A}_1\mathfrak{F} \subseteq \mathfrak{N}\mathfrak{F}$, it follows that $L/K_G \notin \mathfrak{A}_1\mathfrak{F}$, and $H$ is $\mathfrak{A}_1\mathfrak{F}$-abnormal. □

According to Proposition 3.5 (1), every abelian subgroup of a soluble group is $\mathfrak{A}_1\mathfrak{A}$-subnormal. The following example demonstrates that a primary subgroup of nilpotency class at most 2 could be non-$\mathfrak{A}_1\mathfrak{A}$-subnormal.

**Example 3.6.** Let $E_{3^2}$ be the elementary abelian group of order $3^2$. The general linear group GL$(2,3)$ is the automorphism group of $E_{3^2}$. The dihedral subgroup $D$ of order 8 is a subgroup of GL$(2,3)$ and acts irreducibly on $E_{3^2}$. So $G = E_{3^2} \rtimes D$ is contained in the holomorph of $E_{3^2}$. Note $G$ has ID 40 among the groups of order 72 in the GAP SmallGroup library [22]. The Sylow 2-subgroup $D$ of $G$ is a maximal subgroup and $D_G = 1$. Hence $G \in (\mathfrak{A}_1)^3 \setminus \mathfrak{A}_1\mathfrak{A}$ and $D$ is $\mathfrak{A}_1\mathfrak{A}$-abnormal in $G$. It follows that subgroups of nilpotency class 2 could be non-$\mathfrak{A}_1\mathfrak{A}$-subnormal.

**Corollary 3.7.** *Let $\mathfrak{F}$ be a subgroup-closed formation and let $\mathfrak{A}_1\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$. Then* $\mathrm{w}\mathfrak{F} = \mathfrak{S}$.

*Proof.* Every Sylow subgroup of a soluble group is $\mathfrak{A}_1\mathfrak{N}$-subnormal, by Proposition 3.5 (1). Hence $\mathfrak{S} \subseteq \mathrm{w}(\mathfrak{A}_1\mathfrak{N}) \subseteq \mathrm{w}\mathfrak{F}$. The converse is true by Lemma 3.4. □

Substituting $\mathfrak{F} = \mathfrak{A}$ in Proposition 3.5 (2)–(3), we obtain the following:

**Corollary 3.8.** *A subgroup $H$ of a soluble group $G$ is $\mathfrak{A}_1\mathfrak{A}$-subnormal ($\mathfrak{A}_1\mathfrak{A}$-abnormal) if and only if $H$ is $\mathfrak{N}\mathfrak{A}$-subnormal ($\mathfrak{N}\mathfrak{A}$-abnormal).*

**Corollary 3.9.** *Let $\mathfrak{F}$ be a subgroup-closed formation and let $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathfrak{A}$. A subgroup $H$ of a soluble group $G$ is $\mathfrak{F}$-subnormal ($\mathfrak{F}$-abnormal) if and only if $H$ is $\mathfrak{N}\mathfrak{A}$-subnormal ($\mathfrak{N}\mathfrak{A}$-abnormal).*

*Proof.* Suppose that $H$ is an $\mathfrak{F}$-subnormal subgroup of a soluble group $G$. Then $H$ is $\mathfrak{N}\mathfrak{A}$-subnormal, because $\mathfrak{F} \subseteq \mathfrak{N}\mathfrak{A}$. Conversely, assume that $H$ is $\mathfrak{N}\mathfrak{A}$-subnormal in $G$. By Corollary 3.8, $H$ is $\mathfrak{A}_1\mathfrak{A}$-subnormal. Since $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F}$, it implies that $H$ is $\mathfrak{F}$-subnormal in $G$.

Now assume that $H$ is $\mathfrak{F}$-abnormal in $G$. As $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F}$, it follows that $H$ is $\mathfrak{A}_1\mathfrak{A}$-abnormal, and in view of Corollary 3.8, $H$ is $\mathfrak{N}\mathfrak{A}$-abnormal. Conversely, suppose that $H$ is $\mathfrak{N}\mathfrak{A}$-abnormal in $G$. Then $H$ is $\mathfrak{F}$-abnormal, since $\mathfrak{F} \subseteq \mathfrak{N}\mathfrak{A}$. □

**Proposition 3.10.** *If $\mathfrak{F}$ is a subgroup-closed formation and $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$, then* $\mathrm{w}\mathfrak{F} = \mathfrak{N}\mathcal{A}$.

*Proof.* Firstly, we show that $\mathrm{w}\mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. Suppose that it is not true and let $G$ be a group of least order such that $G \in \mathrm{w}\mathfrak{F} \setminus \mathfrak{N}\mathcal{A}$. By hypothesis, $\mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$, it implies that $G^{\mathfrak{F}} \neq 1$. In view of Lemma 3.4, $\mathrm{w}\mathfrak{F}$ is a subgroup-closed soluble formation. Thus for every nontrivial normal subgroup $K$ of $G$, the quotient group $G/K \in \mathrm{w}\mathfrak{F}$ and, by induction, $G/K \in \mathfrak{N}\mathcal{A}$. From Lemma 2.6, it follows that $\mathfrak{N}\mathcal{A}$ is a saturated formation, therefore $G$ is primitive in view of Lemma 2.10. By Lemma 2.11, $G = N \rtimes M$, where $N$ is a unique minimal normal $p$-subgroup of $G$ for some $p \in \pi(G)$ such that $N = C_G(N) = F(G) \in \mathfrak{A}_1$, $M$ is a maximal subgroup of $G$, $M_G = 1$ and $O_p(M) = 1$. We claim that $M \in \mathcal{A}$. Indeed, suppose that in $M$ there is a non-abelian Sylow $q$-subgroup $Q$ for some $q \in \pi(M)$. By induction, $M \in \mathfrak{N}\mathcal{A}$, hence $M/F(M) \in \mathcal{A}$. Since $O_p(M) = 1$, we deduce that $F(M)$ is $p'$-subgroup. If $q = p$, then $Q$ is abelian, a contradiction. So, $q \neq p$. Consider $H = NQ = N \rtimes Q$. If $H < G$, then we have, by induction, $H \in \mathfrak{N}\mathcal{A}$. As $N = C_G(N) = F(H)$, we obtain $H/F(H) \simeq Q \in \mathfrak{A}$, a contradiction. Consequently, $G = N \rtimes Q$. From $G \in \mathrm{w}\mathfrak{F}$, it follows that $Q$ is $\mathfrak{F}$-subnormal in $G$, and so in $G$ there is a maximal subgroup $L$, containing $Q$, such that $G/L_G \in \mathfrak{F}$. Now, $N \subseteq G^{\mathfrak{F}}$ and $G = N \rtimes Q \subseteq G^{\mathfrak{F}}Q \subseteq L$, a contradiction. Thus, $M \in \mathcal{A}$ and $G \in \mathfrak{N}\mathcal{A}$, i.e. $\mathrm{w}\mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$.

To prove the reverse inclusion, we suppose that it is not true and $G$ is a group of least order such that $G \in \mathfrak{N}\mathcal{A} \setminus \mathrm{w}\mathfrak{F}$. Let $N$ be a minimal normal subgroup of $G$ and let $R$ be a non-$\mathfrak{F}$-subnormal Sylow subgroup of $G$. By induction, $G/N \in \mathrm{w}\mathfrak{F}$, therefore $RN/N$ is $\mathfrak{F}$-subnormal in $G/N$ and, by Lemma 2.1 (2), $RN$ is $\mathfrak{F}$-subnormal in $G$. If $RN < G$, then $RN \in \mathrm{w}\mathfrak{F}$ and $R$ is $\mathfrak{F}$-subnormal in $RN$. By Lemma 2.1 (1), $R$ is $\mathfrak{F}$-subnormal in $G$, a contradiction. Now, $G = N \rtimes R$. Since $G \in \mathfrak{N}\mathcal{A}$, it implies that $R$ is abelian and $G \in \mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathrm{w}\mathfrak{F}$, a contradiction. Thus, $\mathfrak{N}\mathcal{A} \subseteq \mathrm{w}\mathfrak{F}$. □

**Lemma 3.11.** *Let $G$ be a soluble group of order $p^n m$, $p$ does not divide $m$. If for every $q \neq p$, a Sylow $q$-subgroup of $G$ is cyclic, then $G \in \mathfrak{N}\mathcal{A}$. In particular, any group of order $p^n q$, where $p$ and $q$ are primes, belongs to $\mathfrak{N}\mathcal{A}$.*

*Proof.* Suppose that $G$ is a counterexample of least order. Since $\mathfrak{N}\mathcal{A}$ is a saturated formation, by Lemma 2.10 and Lemma 2.11, $G$ is primitive and

$$G = N \rtimes M, \quad N = C_G(N) = F(G) = O_r(G),$$
$$r \in \pi(G), \quad M \lessdot G, \quad M_G = \Phi(G) = 1.$$

If $r \neq p$, then $N = G_q$ is cyclic and $G/N$ is abelian, and so $G \in \mathfrak{N}\mathcal{A}$. Suppose that $r = p$. As $O_p(M) = 1$, we conclude that $F(M)$ is cyclic $p'$-subgroup. Now $M/F(M)$ is abelian by [10, 2.16], it follows that a Sylow $p$-subgroup of $M$ is abelian. Thus, $M \in \mathcal{A}$ and $G \in \mathfrak{N}\mathcal{A}$. □

**Example 3.12.** In the group $G = E_{3^2} \rtimes D$ from example 3.6, the Sylow 2-subgroup $D$ is $\mathfrak{A}_1\mathfrak{A}$-abnormal. Hence $G \notin \mathrm{w}(\mathfrak{A}_1\mathfrak{A}) = \mathfrak{N}\mathcal{A}$. Thus in Lemma 3.11, the condition for Sylow $2'$-subgroups to be cyclic cannot be replaced by the condition to be abelian.

### Proof of Theorem A

(1) If every Sylow subgroup of $G$ is $\mathfrak{F}$-subnormal, then $G \in \mathrm{w}\mathfrak{F}$. By Proposition 3.10, $\mathrm{w}\mathfrak{F} = \mathfrak{N}\mathcal{A}$, it follows that $G^{\mathcal{A}}$ is nilpotent. Conversely, if $G^{\mathcal{A}}$ is nilpotent, then $G \in \mathfrak{N}\mathcal{A} = \mathrm{w}\mathfrak{F}$ and every Sylow subgroup of $G$ is $\mathfrak{F}$-subnormal.

(2) We begin by proving $\mathfrak{N}\mathcal{A} \cap \mathfrak{N}^2 = \mathfrak{N}\mathfrak{A}$. It is clear that $\mathfrak{N}\mathfrak{A} \subseteq \mathfrak{N}\mathcal{A} \cap \mathfrak{N}^2$. To prove the reverse inclusion, we suppose that it is not true and $G$ is a group of least order such that $G \in (\mathfrak{N}\mathcal{A} \cap \mathfrak{N}^2) \setminus \mathfrak{N}\mathfrak{A}$. If $K$ is a nontrivial normal subgroup of $G$, then $G/K \in \mathfrak{N}\mathfrak{A}$ by induction. Consequently, $G$ is primitive by Lemma 2.10, and in view of Lemma 2.11, $G = F(G) \rtimes M$. Since $G \in (\mathfrak{N}\mathcal{A} \cap \mathfrak{N}^2)$, we deduce $G/F(G) \simeq M \in \mathcal{A} \cap \mathfrak{N} = \mathfrak{A}$. Hence $G \in \mathfrak{N}\mathfrak{A}$, a contradiction. Thus, we have $\mathrm{w}\mathfrak{F} \cap \mathfrak{N}^2 = \mathfrak{N}\mathcal{A} \cap \mathfrak{N}^2 = \mathfrak{N}\mathfrak{A}$.

If $G \in \mathrm{w}\mathfrak{F}$, then $G$ is soluble by Lemma 2.13. Let $H$ be a metanilpotent subgroup of $G$. By Lemma 3.4, $H \in \mathrm{w}\mathfrak{F} \cap \mathfrak{N}^2$. Since $\mathrm{w}\mathfrak{F} \cap \mathfrak{N}^2 = \mathfrak{N}\mathfrak{A}$, it implies that the derived subgroup of $H$ is nilpotent. To prove the converse, we suppose that it is not true and there is a soluble group $G \notin \mathrm{w}\mathfrak{F}$ such that its metanilpotent subgroup has the nilpotent derived subgroup. Let $H$ be a minimal non-$\mathrm{w}\mathfrak{F}$-subgroup in $G$. Since $\mathrm{w}\mathfrak{F} = \mathfrak{N}\mathcal{A}$, from Lemma 2.8 we conclude that $H$ is metanilpotent. By the choice of $G$, $H \in \mathfrak{N}\mathfrak{A} \subseteq \mathfrak{N}\mathcal{A}$, a contradiction. Thus, $G \in \mathrm{w}\mathfrak{F}$. Theorem A is proved.

Since $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{A}^2 \subseteq \mathfrak{N}\mathfrak{A} \subseteq \mathfrak{N}\mathcal{A}$, we obtain the following:

**Corollary 3.13.** *We have* $\mathrm{w}(\mathfrak{A}_1\mathfrak{A}) = \mathrm{w}(\mathfrak{A}^2) = \mathrm{w}(\mathfrak{N}\mathfrak{A}) = \mathrm{w}(\mathfrak{N}\mathcal{A}) = \mathfrak{N}\mathcal{A}$.

**Corollary 3.14.** *Let $\mathfrak{F}$ be a subgroup-closed formation and let $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. Then $\mathrm{w}\mathfrak{F}$ is a soluble saturated subgroup-closed formation.*

*Proof.* By Proposition 3.10, we have $\mathrm{w}\mathfrak{F} = \mathfrak{N}\mathcal{A}$. The formation $\mathfrak{N}\mathcal{A}$ is soluble and subgroup-closed by Lemma 3.4, and saturated by Lemma 2.6. □

**Corollary 3.15.** *Let $\mathfrak{F}$ be a subgroup-closed formation and let $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. If $G \in \mathrm{w}\mathfrak{F}$, then every nilpotent subgroup of $G$ is $\mathfrak{F}$-subnormal.*

*Proof.* We use induction on $|G|$. Suppose that $G \in \mathrm{w}\mathfrak{F}$ is a group of least order that contains a nilpotent non-$\mathfrak{F}$-subnormal subgroup $H$. Let $N$ be a minimal normal subgroup of $G$. In view of Lemma 3.4, $HN \in \mathrm{w}\mathfrak{F}$. Since $HN$ is metanilpotent, it follows that $HN \in \mathfrak{N}\mathfrak{A}$ by Theorem A (2) and $H$ is $\mathfrak{N}\mathfrak{A}$-subnormal in $HN$.

As a consequence of Corollary 3.8, $H$ is $\mathfrak{A}_1\mathfrak{A}$-subnormal in $HN$, and so $H$ is $\mathfrak{F}$-subnormal in $HN$, since $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F}$. By induction, $HN/N$ is $\mathfrak{F}$-subnormal in $G/N$, and by Lemma 2.1 (2), $HN$ is $\mathfrak{F}$-subnormal in $G$. Finally, from part (1) of Lemma 2.1 we conclude that $H$ is $\mathfrak{F}$-subnormal in $G$. □

**Remark 3.16.** For any subgroup-closed formation $\mathfrak{F}$ such that $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$, we have $\mathrm{w}\mathfrak{F} = \mathfrak{N}\mathcal{A}$. Therefore Lemma 2.8 contains the description of minimal non-$\mathrm{w}\mathfrak{F}$-groups.

## 4 Groups with $\mathfrak{F}$-subnormal and $\mathfrak{F}$-abnormal Sylow subgroups

A Carter subgroup is a nilpotent self-normalizing subgroup. A Gaschütz subgroup is a supersoluble subgroup $H$ such that $|L : K|$ is not prime for all subgroups $K$ and $L$, $H \leq K < L \leq G$.

**Lemma 4.1.** *Let $\mathfrak{F}$ be a soluble subgroup-closed formation. If every Sylow subgroup of a group $G$ is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal, then $G$ is soluble.*

*Proof.* If there is an $\mathfrak{F}$-subnormal Sylow subgroup in $G$, then by Lemma 2.13, $G$ is soluble. Assume that every Sylow subgroup of $G$ is $\mathfrak{F}$-abnormal. Then each one is self-normalizing in view of Lemma 2.2 (1), and so is a Carter subgroup. By Vdovin's theorem [21], Carter subgroups are conjugate, therefore $G$ is primary and soluble. □

**Proposition 4.2.** *Let $\mathfrak{F}$ be a formation and let $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. If every Sylow subgroup of a group $G$ is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal, then either $G \in \mathfrak{N}\mathcal{A}$ or the following statements hold:*

(1) *Only one of the Sylow subgroups in $G$ is $\mathfrak{F}$-abnormal; let $P$ be such a Sylow $p$-subgroup of $G$.*

(2) *$G$ is soluble, $P$ is a non-abelian Cater and Gaschütz subgroup.*

(3) *$G_{p'} \in \mathrm{w}\mathfrak{F}$ and $G_{p'} \leq G^{\mathfrak{N}} = G^{\mathfrak{U}}$.*

*Proof.* If every Sylow subgroup of $G$ is $\mathfrak{F}$-subnormal, then $G \in \mathfrak{N}\mathcal{A}$ by Proposition 3.10. Assume that $G \notin \mathfrak{N}\mathcal{A}$. In view of Proposition 3.10, in $G$ there is an $\mathfrak{F}$-abnormal Sylow $p$-subgroup $P$ for a prime $p$. By Lemma 2.2 (1), $P$ is self-normalizing, and so a Carter subgroup of $G$. By Vdovin's theorem [21], Carter subgroups are conjugate, therefore every Sylow $r$-subgroup of $G$, $r \neq p$, is different from its normalizer and $\mathfrak{F}$-subnormal in $G$. From Lemma 4.1, we conclude that $G$ is soluble and $G_{p'} \in \mathrm{w}\mathfrak{F}$. If $P$ is abelian, then $P$ is $\mathfrak{A}_1\mathfrak{A}$-subnormal by Proposition 3.5 (1), and so $\mathfrak{F}$-subnormal, this contradicts the $\mathfrak{F}$-abnormality of $P$. Therefore $P$ is non-abelian.

To prove that $P$ is a Gaschütz subgroup, we suppose that this is not true and that in $G$ there are subgroups $K$ and $L$ such that

$$P \le K \lessdot L \le G, \quad |L:K| = r \in \mathbb{P}.$$

Then $L/K_L$ is a primitive group and, by Lemma 2.11,

$$L/K_L = N/K_L \rtimes K/K_L, \quad N/K_L = C_{L/K_L}(N/K_L) = F(L/K_L),$$
$$|N/K_L| = |L:K| = r \in \mathbb{P}, \quad N/K_L \in \mathfrak{A}_1,$$
$$N_{L/K_L}(N/K_L)/C_{L/K_L}(N/K_L) = (L/K_L)/(N/K_L) \in \mathfrak{A}$$

in view of [10, Theorem 2.16 (3)]. Hence $L/K_L \in \mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F}$, this contradicts the $\mathfrak{F}$-abnormality of $P$. Thus $P$ is a Gaschütz subgroup of $G$.

As a Gaschütz subgroup is a $\mathfrak{U}$-projector [10, Theorem 5.29], we obtain

$$G = G^{\mathfrak{U}}P, \quad G/G^{\mathfrak{U}} \simeq P/P \cap G^{\mathfrak{U}} \in \mathfrak{N}, \quad G^{\mathfrak{N}} \le G^{\mathfrak{U}}.$$

Since $\mathfrak{N} \subseteq \mathfrak{U}$, we conclude $G^{\mathfrak{U}} \le G^{\mathfrak{N}}$ and $G^{\mathfrak{U}} = G^{\mathfrak{N}}$. From $G = G^{\mathfrak{U}}P$, it follows that $G_{p'} \le G^{\mathfrak{U}}$. □

A group with a normal Sylow $p$-subgroup is called $p$-closed.

**Lemma 4.3.** *Let $\mathfrak{F}$ be a formation and let $\mathfrak{A}_1\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{N}\mathcal{A}$. Assume that $G$ is a $\{p,q\}$-group with an $\mathfrak{F}$-subnormal Sylow $p$-subgroup $P$ and an $\mathfrak{F}$-abnormal Sylow $q$-subgroup $Q$, $p \neq q$. If $Q' \le Z(Q)$, then $P$ is normal in $G$.*

*Proof.* By Lemma 2.12, all $p$-subgroups are $\mathfrak{F}$-subnormal in $G$. Suppose that $G$ is a group of least order with a non-normal Sylow subgroup $P$. Let $K$ be a nontrivial normal subgroup of $G$. In view of the properties of Sylow subgroups, Lemma 2.1 (2) and Lemma 2.4 (2), $PK/K$ is normal in $G/K$ by induction, i.e. $G/K$ is $p$-closed. Hence, by Lemma 2.10, $G$ is primitive and, by Lemma 2.11, $G = N \rtimes M$, where $N$ is a unique minimal normal $r$-subgroup of $G$ such that $N = C_G(N) = F(G) \in \mathfrak{A}_1$, $M$ is a maximal subgroup of $G$ with trivial core and $O_r(M) = 1$. If $r = p$, then $Q \le M$ and, by induction, $M$ is $p$-closed, but $O_p(M) = 1$, a contradiction. Consequently, $r = q$ and $N$ is a proper subgroup in $Q$ in view of Lemma 2.2 (1). By induction, we have $PN/N \lhd G/N$, and so $H = PN = N \rtimes P$ is a proper normal subgroup of $G$. From Lemma 2.1 (5) we conclude that $P$ is $\mathfrak{F}$-subnormal in $H$, and $N$ is also $\mathfrak{F}$-subnormal in $H$ in view of Lemma 2.12. Consequently, $H \in \mathrm{w}\mathfrak{F} = \mathfrak{N}\mathcal{A}$ by Proposition 3.10. It follows from $C_G(N) = N$ that $P_H = 1$ and $F(H) = N$, and so $H/F(H) \simeq P \in \mathfrak{A}$. By induction, $M = P \rtimes Q_1$, where $Q = N \rtimes Q_1$. Since $Q' \le Z(Q) \le C_G(N) = N$, we get $Q_1 \in \mathfrak{A}$. Thus, $G \in \mathfrak{N}\mathcal{A}$ and $Q$ is $\mathfrak{F}$-subnormal in $G$ by Proposition 3.10, a contradiction. □

**Remark 4.4.** In the symmetric group $S_4$ of degree 4, the Sylow 2-subgroup $D$ is abnormal and has nilpotency class 2. A Sylow 3-subgroup $Z$ is not normal but $\mathfrak{A}_1\mathfrak{A}$-subnormal, because

$$Z \leq A_4 \lhd S_4, \quad A_4 \in \mathfrak{A}_1\mathfrak{A}.$$

Thus in Lemma 4.3 we cannot replace the $\mathfrak{A}_1\mathfrak{A}$-abnormality by abnormality of Sylow subgroups.

**Proof of Theorem B**

Suppose that every Sylow subgroup of a group $G$ is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal of nilpotency class at most 2. By Lemma 4.1, $G$ is soluble. If every Sylow subgroup of $G$ is $\mathfrak{F}$-subnormal, then $G \in \mathrm{w}\mathfrak{F} = \mathfrak{N}\mathcal{A}$ in view of Proposition 3.10.

Assume that $G \notin \mathfrak{N}\mathcal{A}$. Then in $G$, there is an $\mathfrak{F}$-abnormal Sylow $p$-subgroup $P$ for some $p \in \pi(G)$ of nilpotency class at most 2. By Proposition 4.2, $P$ is a non-abelian Carter and Gaschütz subgroup and $G^{\mathfrak{N}} = G^{\mathfrak{U}}$. In view of [10, Theorems 5.27 and 5.29] and Lemma 2.9, $G = G^{\mathfrak{N}}P = G^{\mathfrak{U}}P$. Since Carter subgroups are conjugate [21], it implies that every Sylow $q$-subgroup $G_q$ of $G$, $q \neq p$, is $\mathfrak{F}$-subnormal in $G$. Consider a Hall $\{p,q\}$-subgroup $H = PG_q$ of $G$. It follows from Lemma 4.3 that $G_q$ is normal in $H$ and $P \leq N_G(G_q)$. Since $q$ is arbitrary, we obtain $P \leq N_G(G_{p'})$ and the Hall $p'$-subgroup $G_{p'}$ of $G$ is normal in $G$. Consequently,

$$G = G_{p'} \rtimes P = G^{\mathfrak{N}}P = G^{\mathfrak{U}}P, \quad G_{p'} = G^{\mathfrak{N}} = G^{\mathfrak{U}}.$$

By Proposition 3.10, $G_{p'} \in \mathfrak{N}\mathcal{A}$.

Conversely, if $G \in \mathfrak{N}\mathcal{A}$, then every Sylow subgroup of $G$ is $\mathfrak{F}$-subnormal by Proposition 3.10. Now assume that $G = G^{\mathfrak{N}} \rtimes P$, where $P$ is a non-abelian $\mathfrak{F}$-abnormal Sylow $p$-subgroup of $G$ for some element $p \in \pi(G)$ and a Carter and Gaschütz subgroup, $P' \leq Z(P)$ and $G^{\mathfrak{N}} = G^{\mathfrak{U}} \in \mathfrak{N}\mathcal{A}$. Let $G_r$ be a Sylow $r$-subgroup of $G$, $r \in \pi(G)$. If $r = p$, then $G_r$ and $P$ are conjugate, so $G_r$ is $\mathfrak{F}$-abnormal in $G$. If $r \neq p$, then $G_r \leq G^{\mathfrak{N}} \in \mathfrak{N}\mathcal{A}$. In view of Proposition 3.10, $G_r$ is $\mathfrak{F}$-subnormal in $G^{\mathfrak{N}}$, consequently, $G_r$ is $\mathfrak{F}$-subnormal in $G$. Theorem B is proved.

**Example 4.5.** Let $E_{2^4}$ be the elementary abelian group of order 16. The general linear group $\mathrm{GL}(4,2) \simeq A_8$ is the automorphism group of $E_{2^4}$ and contains $H = E_{3^2} \rtimes D$ (see Example 3.6). The group $G = E_{2^4} \rtimes H$ is a subgroup of the holomorph and has ID 157849 among the groups of order 1152 in the GAP SmallGroup library [22]. Besides,

$$Q = E_{2^4} \rtimes D \lessdot G, \quad Q_G = E_{2^4} = F(G), \quad G/Q_G \simeq H \notin \mathfrak{A}_1\mathfrak{A}.$$

Hence the Sylow 2-subgroup $Q$ is $\mathfrak{A}_1\mathfrak{A}$-abnormal in $G$. The Sylow 3-subgroup $P = E_{3^2}$ is not normal but $\mathfrak{A}_1\mathfrak{A}$-subnormal in $G$, since

$$P \leq E_{2^4} \rtimes P \lhd G, \quad E_{2^4} \rtimes P \in \mathfrak{A}_1^2.$$

Thus we cannot omit the restriction on the nilpotency class of $\mathfrak{F}$-abnormal Sylow subgroups in Theorem B.

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Received June 27, 2017; revised October 5, 2017.

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