www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 9 No. 3 (2020), pp. 139-142.

www.ui.ac.ir

FINITE GROUPS WITH SEMINORMAL OR ABNORMAL SYLOW SUBGROUPS

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Abstract. Let G be a finite group in which every Sylow subgroup is seminormal or abnormal. We prove that G has a Sylow tower. We establish that if a group has a maximal subgroup with Sylow subgroups under the same conditions, then this group is soluble.

1. Introduction

All groups in this paper are finite.

A subgroup A is said to be *seminormal* in a group G if there is a subgroup B such that $G = AB$ and AB_1 is a proper subgroup in G for every proper subgroup B_1 of B. Groups with certain seminormal subgroups were investigated by many authors, see, for example, [1, 2, 3, 4, 5, 6, 7, 8]. In particular, a group with seminormal Sylow subgroups is supersoluble [5, 6, 8].

A subgroup H of a group G is abnormal if $x \in \langle H, H^x \rangle$ for every $x \in G$. In the symmetric group S_4 of degree 4, a Sylow 2-subgroup is both seminormal and abnormal.

In this paper, we investigate groups in which every Sylow subgroup is seminormal or abnormal. We prove that such group has a Sylow tower and a normal supersoluble subgroup of primary index.

2. Preliminaries

We write $A \leq B$ if A is a subgroup of a group B, and write $A < B$ if A is a proper subgroup of B. Let G be a group. We use $\pi(G)$ to denote the set of all prime divisors of $|G|$. If $p \in \pi(G)$, then G_p,

Communicated by Evgeny Vdovin

MSC(2010): Primary: 20D20; Secondary: 20D40.

Keywords: Finite group, Sylow subgroup, abnormal subgroup, seminormal subgroup.

Received: 19 August 2018, Accepted: 17 October 2018.

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http://dx.doi.org/10.22108/ijgt.2018.112602.1500

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$O_p(G)$ and $G_{p'}$ represent a Sylow p-subgroup of G, the largest normal p-subgroup of G and a Hall p'-subgroup of G (if it exist), respectively. A group G is p-closed if $G_p = O_p(G)$, and p-nilpotent if there is $G_{p'}$ which is normal in G. We use $Z(G)$, G' and $\Phi(G)$ to denote the centre, derived subgroup and Frattini subgroup of G, respectively. The semidirect product of a normal subgroup A and a subgroup B is denoted by $A \rtimes B$. The symbol □ indicates the end of the proof.

We need the following properties of seminormal and abnormal subgroups.

Lemma 2.1 ([7, Lemma 2])**.** (1) *If H is a seminormal subgroup of a group G and $H \leq K \leq G$, then H is seminormal in K.*

(2) *If H is a seminormal subgroup of a group G and N is a normal subgroup of G, then HN/N is seminormal in G/N.*

Lemma 2.2 ([9, I.6])**.** (1) *If A is an abnormal subgroup of a group G and $A \leq B \leq G$, then A is abnormal in B and $N_G(A) = A$.*

(2) *If A is an abnormal subgroup of a group G and $A \leq B \leq G$, then B is abnormal in G and $N_G(B) = B$.*

(3) *If A is an abnormal subgroup of a group G and N is a normal subgroup of G, then AN/N is abnormal in G/N.*

Lemma 2.3 ([7, Lemma 10])**.** *If A is a seminormal 2-nilpotent subgroup of a group G, then A^G is soluble.*

3. Groups with seminormal and abnormal Sylow subgroups

Example 3.1. *In the alternating group A_4 of degree 4, a Sylow 2-subgroup is normal, a Sylow 3-subgroup is abnormal.*

Example 3.2. *In any minimal non-nilpotent group $S = G_p \rtimes G_q$ with $\Phi(S) = 1$, G_p is normal, G_q is abnormal,* [10, IV.5.4]*.*

Lemma 3.3 ([6, Corollary 6])**.** *Let G be a group and let $r \in \pi(G)$. If Sylow p-subgroups of G are seminormal for all $p \in \pi(G) \setminus \{r\}$, then G is soluble and r-supersoluble. In particular, if all Sylow subgroups of G is seminormal, then G supersoluble.*

Lemma 3.4. *Let $G = G_pG_q$. If G_p is seminormal in G and $G_q = N_G(G_q)$, then G_p is normal in G.*

Proof. Suppose that G is not p-closed and apply induction on the order of G. By Lemma 2.1 and Lemma 2.2, in G all non-trivial quotient groups are p-closed, therefore G is primitive [9, A.15.1]:

$$O_p(G) = \Phi(G) = 1, \; N = O_q(G) = C_G(N) < G_q.$$

Choose $N_1 \leq N \cap Z(G_q) \neq 1$, $|N_1| = q$. Since G_p is seminormal in G, we get G_pN_1 is a subgroup and N_1 is normal in N_1G_p. Hence $N_G(N_1) \geq \langle G_q, G_p \rangle = G$ and N_1 is normal in G. So, $N_1 = N = G_q$, a contradiction. □

http://dx.doi.org/10.22108/ijgt.2018.112602.1500

Example 3.5. *In the symmetric group S_4, there is no normal Sylow subgroup: a Sylow 2-subgroup is seminormal and abnormal, a Sylow 3-subgroup is neither seminormal nor abnormal. Hence in Lemma 3.4* none of conditions (seminormality of a Sylow subgroup and abnormality of other one) cannot be omitted.

Theorem 3.6. *Let G be a group in which every Sylow subgroup is seminormal or abnormal. Then G is supersoluble or $G = G_{r'} \rtimes G_r$ for some $r \in \pi(G)$, $G_{r'}$ is supersoluble, $G_r = N_G(G_r)$.*

Proof. If all Sylow subgroups of G are seminormal, then G is supersoluble by Lemma 3.3. Suppose that in G there is a non-seminormal Sylow r-subgroup G_r for some $r \in \pi(G)$. By the hypothesis, G_r is abnormal and $G_r = N_G(G_r)$ by Lemma 2.2. By E. P. Vdovin theorem [11], $G_p < N_G(G_p)$ for all $p \in \pi(G) \setminus \{r\}$. Hence all such G_p are seminormal in G, and G is soluble by Lemma 3.3. Now, $G = G_{r'}G_r$ and $G_{r'}$ is supersoluble in view of Lemma 2.2 and Lemma 3.3. If G_pG_r is a biprimary Hall $\{p, r\}$-subgroup of G, then G_p is seminormal in G_pG_r by Lemma 2.1 and normal in G_pG_r by Lemma 3.4. Therefore $G_{r'}$ is normal in G. □

Corollary 3.7. *If every Sylow subgroup of a group G is seminormal or abnormal, then G has a Sylow tower and the nilpotent length of G is not more than 3.*

Example 3.8. *Example* 3.1 and Example 3.2 show that G can have a Sylow tawer of any type.

Remark 3.9. *In* [6, Theorem 3]*, it was proved that a group G is soluble if every non-cyclic Sylow subgroup of G is seminormal. A group with abnormal non-cyclic Sylow subgroup can be insoluble, for example,* $\mathrm{PSL}(2, 17)$ *and* $\mathrm{PSL}(2, 31)$*. In these groups, Sylow subgroups of odd orders are cyclic, Sylow 2-subgroups are maximal and therefore abnormal.*

Theorem 3.10. *Let M be a maximal subgroup of a group G and let P be a Sylow 2-subgroup in M. Assume that every Sylow subgroup of M is seminormal in G or abnormal in G. If $P' \le Z(P)$ for abnormal P, then G is soluble.*

Proof. We use induction on the order of G. In view of Lemma 2.1 and Lemma 2.2, every Sylow subgroup of M is seminormal in M or abnormal in M.

Case 1. All Sylow subgroup of M is seminormal in G.

Choose a Sylow subgroup R of M. By the hypothesis, R is seminormal in G. According to Lemma 2.3, R^G is soluble. Let H be the product of all R^G, where R runs all Sylow subgroup of M. Hence H is soluble, $M \le H$ and H is normal in G. If $H = G$, then G is soluble. If $M = H$, then $|G : M|$ is a prime, and G is soluble.

Case 2. There is a Sylow r-subgroup R of M that is not seminormal in G.

By the hypothesis, R is abnormal in G, $R = N_G(R)$ by Lemma 2.2, and R is a Sylow subgroup of G. If $R = M$, then G is soluble [10, IV.7.4]. Therefore $M = R \ltimes M_{r'}$ by Theorem 3.6, $M_{r'} \ne 1$ is a supersoluble subgroup of M. Let Q be a Sylow q-subgroup of M, $q \ne r$. If Q is abnormal in G, then $Q = N_G(Q)$ by Lemma 2.2, and from E. P. Vdovin theorem [11] it follows that Q and R are conjugate, a contradiction. Hence Q is not abnormal in G. By the hypothesis, Q is seminormal in G. In view of Lemma 2.3, Q^G is soluble. If U is the product of all Q^G, where Q runs all Sylow subgroups of $M_{r'}$, then $M_{r'} \leq U$, U is soluble and normal in G. If $MU = G$, then G is soluble. If $U = M$, then $|G : M|$ is a prime, and G is soluble. Suppose that $U < M$. Then $UR = M$, a maximal subgroup $M/U \cong R/R \cap U$ of G/U is a Sylow r-subgroup which is abnormal in G/U according to Lemma 2.2. By the hypothesis, $R' \leq Z(R)$, it implies that $(M/U)' \leq Z(M/U)$. From [10, IV.7.4] it follows that G/U is soluble, hence G is soluble. □

Corollary 3.11. *Let M be a maximal subgroup of a group G. If M is of odd order and every Sylow subgroup of M is seminormal in G or abnormal in G, then G is soluble.*

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http://dx.doi.org/10.22108/ijgt.2018.112602.1500