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Right-Side Hyperbolic Operators

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Abstract: In the paper a new class of linear operators was introduced: linear operator B is said to be *right-side hyperbolic*, if operators $B-\lambda I$ are right-sided invertible for any λ from a neighborhood of the unit circle and moreover one can specify *right-side resolvent* $R_r(B;\lambda)$ namely a family of right inverse to $B-\lambda I$ analytic in λ. In the paper general form of right-side resolvents is given. We also discuss a distinguishes with the hyperbolic case.

Keywords: Resolvent, Riesz projection, one-side invertibility.

1 Introduction

One of the central problems of operator theory consists in obtaining invertibility conditions and explicit construction of the inverse for various classes of operators. Important variant of this problem is a description of the spectrum $\sigma(B)$ of an operator B, i.e. searching for the conditions of invertibility for operator $B-\lambda I$. Mostly detailed results in this direction are obtained for operator of hyperbolic class for which the resolvent $R(B;\lambda) := (B-\lambda I)^{-1}$ can be expressed explicitly.

Along with the investigation of invertibility itself, the study of right-side invertibility is of a considerable interest since this property guarantees existence of solution for corresponding nonhomogeneous equation as well as construction of right-side inverse operator which is equivalent to resolution of equation in question. Note that for operator $B-\lambda I$ this problem makes sense for value of λ belonging to the spectrum $\sigma(B)$.

In the present paper we introduce a class of right-side hyperbolic operators and give general form of right-side resolvent. The goal of the paper is to specify the properties of such operators which are similar to those of hyperbolic ones or are essentially different.

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2 Hyperbolic operators

Let a bounded linear operator B acts in Banach space F. Suppose that a simple closed contour Γ does not intersect with the spectrum $\sigma(B)$ of operator B. It is well known [11, chapter XI] that formula

$$P = -\frac{1}{2\pi i}\int_{\Gamma} R(B;\lambda)d\lambda$$

gives a Riesz projection commuting with B and carrying out a splitting $F = F^+ \bigoplus F^-$ into a direct sum of subspaces invariant with respect to B, where

$$F^+ = ImP,\ \ F^- = Im(I-P) = \ker P.$$

Moreover operator B is decomposed into a direct sum of operators $B = B^+ \bigoplus B^-$, acting in the corresponding subspaces so that the spectrum of the operator B^+ is the part of $\sigma(B)$ contained inside the contour Γ, while the spectrum of B^- is the part of $\sigma(B)$ located outside contour Γ.

Operator B is called *hyperbolic*, if $\sigma(B)\cap\mathbb{S}^1 = \emptyset$, where $\mathbb{S}^1$ is the unit circle:

$$\mathbb{S}^1 = \{\lambda : |\lambda| = 1\},$$

i.e operators $B - \lambda I$ are invertible provided that $|\lambda| = 1$.

Hyperbolic operators used in different applications, in particular, these operator appear naturally in dynamical system theory [5,7,10]. For hyperbolic operator its Riesz projection is given by the expression

$$P = -\frac{1}{2\pi i}\int_{|\lambda|=1} R(B;\lambda)d\lambda. \tag{1}$$

and for the corresponding operators one has

$$r(B^+) < 1,\ \ r((B^-)^{-1}) < 1,$$

where $r(B)$ is the spectral radius of B.

Provided that operator B is invertible its resolvent can be represented in a neighborhood of the unit circle in the form of the Laurent series

$$(B-\lambda I)^{-1} = \sum_{k=0}^{+\infty}\lambda^k B^{-k-1}(I-P) - \sum_{-\infty}^{-1}\lambda^k B^{-k+1}P. \tag{2}$$

In the hyperbolic case the subspaces F^+ (and similarly F^-) can be characterized by different way as follows:

1) $F^+ = \{u \in F : \lim_{n\to+\infty}\|B^n u\| = 0\}$;

2) $F^+ = \{u \in F : \overline{\lim}_{n\to+\infty}\|B^n u\|^{1/n} < 1\} = \{u : r(u) < 1\}$;

3) there exist constants $C > 0$ and $\gamma < 1$ such that

$$F^+ = \{u \in F : \|B^n u\| \le C\gamma^n\|u\| \ \text{ for } \ n \ge 0\};$$

and moreover F^+ is the maximal of subspaces indicated in the right-hand-side.

3 Right-side hyperbolic operators

Operator B is to be called *right-side hyperbolic*, if operators $B-\lambda I$ are right-sided invertible for any λ from a neighborhood of the unit circle and moreover one can specify *right-side resolvent* $R_r(B;\lambda)$ namely a family of right inverse to $B-\lambda I$ analytic in λ. Bellow we assume that operator B is invertible.

Right-side invertibility condition as well as construction of some right-side resolvents for certain concrete classes of operator was obtained in [2,4,8,9].

Similarly to the hyperbolic case can be obtained the general form of right-side resolvent.

Theorem 1 *Any the right-side resolvent $R_r(B;\lambda)$ for a right-hyperbolic operator B can be represented in a neighborhood of the unit circle in the form of the Laurent operator series*

$$R_r(B;\lambda) = \sum_{k=0}^{+\infty} \lambda^k B^{-k-1}(I-P) - \sum_{-\infty}^{-1} \lambda^k B^{-k+1} P, \tag{3}$$

where operator P is given by the same formula as the Riesz projection:

$$P = -\frac{1}{2\pi i}\int_{\mathbb{S}^1} R_r(B;\lambda)\, d\lambda. \tag{4}$$

Proof. By definition

$$R_r(B;\lambda) = \sum_{-\infty}^{+\infty} A_k \lambda^k.$$

Then

$$-\frac{1}{2\pi i}\int_{\mathbb{S}^1} R_r(B;\lambda)\, d\lambda = A_{-1} := P.$$

From equality $(B-\lambda I)R_r(B;\lambda) = I$ we obtain that

$$BA_k - A_{k-1} = \begin{cases} 0, & \text{for } k \neq 0, \\ I, & \text{for } k = 0. \end{cases}$$

Therefore $A_0 = B^{-1}(I-P)$, $A_1 = B^{-1}A_0 = B^{-2}(I-P)$ and

$$A_k = B^{-k-1}(I-P) \text{ for } k \geq 0.$$

Similarly $A_{-2} = BA_{-1} = BP$, $A_{-3} = BA_{-2} = B^2 P$ and

$$A_k = B^{-k+1} P \text{ for } k < -1. \blacktriangleleft$$

It follows for the theorem 1 that in order to construct a right-side resolvent it is sufficient to obtain corresponding operator P. Let us consider properties of an operator P which are needed in question under consideration.

By condition, the series from (3) converge at an annulus U of the form

$$U = \{\lambda : r^- \le |\lambda| \le r^+\}, \quad \text{where} \quad r^- < 1 < r^+.$$

It follows from this that

$$\overline{\lim}_{k\to\infty} \|B^k P u\|^{1/k} \le r^- < 1, \tag{5}$$

$$\overline{\lim}_{k\to+\infty} \|B^{-k}(I-P)u\|^{1/k} \le \frac{1}{r^+} < 1\}. \tag{6}$$

Let us consider vector subspaces

$$F^+ = \{u \in F : \overline{\lim}_{n\to+\infty} \|B^n u\|^{1/n} < 1\},$$

$$F^- = \{u \in F : \overline{\lim}_{n\to\infty} \|[B^{-1}]^n u\|^{1/n} < 1\}.$$

Operator B is hyperbolic if and only if these subspace are closed and the space F is decomposed into direct sum:

$$F = F^+ \bigoplus F^-.$$

If B is right-side hyperbolic from conditions (5) and (6) follows

$$ImP \subset F^+,\ Im(I-P) \subset F^-. \tag{7}$$

and, in particular,

$$F^+ + F^- = F. \tag{8}$$

We remark that sum in (8) is not direct, but only algebraic.

Lemma 1. *If condition (8) holds, then $Im(B-\lambda I) = F$ for any λ such that $|\lambda| = 1$. If F is a Hilbert space, then operators $B - \lambda I$ are right-side invertible for any λ from a neighborhood of the unit circle.*

If there exists a bounded linear operator P such that inclusions (7) hold, then series from (3) converge for any $f \in F$, expression (3) gives one of the right-side resolvent and B is right-side hyperbolic operator.

Proof. Let $f \in F$. Under condition (8) there exists a decomposition

$$f = f^+ + f^-,\ f^+ \in F^+,\ f^- \in F^-. \tag{9}$$

Then the series

$$\sum_{k=0}^{+\infty} \lambda^k B^{-k-1} f^- := u^+$$

and

$$-\sum_{-\infty}^{-1} \lambda^k B^{-k+1} f^+ := u^-$$

converge for $|\lambda| = 1$ and vector $u = u^+ + u^-$ is a solution of the equation

$$(B - \lambda I)u = f.$$

This means that $Im(B - \lambda I) = F$.

If the (closed) subspaces $\ker(B - \lambda I)$ are complimented, it follows from this, that operators $B - \lambda I$ are right-side invertible for $|\lambda| = 1$. In Hilbert space all closed subspaces are complimented and operators $B - \lambda I$ are right-side invertible for $|\lambda| = 1$.

It is well known, that the set of λ, for which operators $B - \lambda I$ are right-side invertible, is an open set on $\mathbb{C}$. Therefore operators $B - \lambda I$ are right-side invertible for any λ from a neighborhood of the unit circle.

In particular, from inclusions (7) follows condition (8) and operator P gives decomposition (9), where $f^+ = Pf$, $f^- = (-P)$. ◄

Therefore in order to construct a right-side resolvent we need to construct an operator P such that inclusions (7) hold.

4 Distinguishes to hyperbolic case

Lemma 3. *Let there exists an operator P such that (7) hold and B commuting with B, then operator B is hyperbolic and P is the Riesz projection.*

This lemma gives first distinguish: in contrast to hyperbolic case operator P generating right-side resolvent can not commute with B.

Let us demonstrate another possible distinguishes by example.

Let

$$F = l_2(\mathbb{Z}) = \{u = (u(k)) : k \in \mathbb{Z}, u(k) \in \mathbb{C}, \sum |u(k)|^2 < +\infty\}.$$

We shell consider a weighted shift operator B acting in the Hilbert space $l_2(\mathbb{Z})$ by expression

$$(Bu)(k) = a(k)u(k+1),\ u \in l_2(\mathbb{Z}),$$

where

$$a(k) = \begin{cases} 2, & k \geq 0, \\ 1/2, & k < 0. \end{cases}$$

It is known that [2,3]

$$\sigma(B) = \{\lambda : 1/2 \leq |\lambda| \leq 2\}$$

and that operators $B - \lambda I$ are right-side invertible under condition [4]

$$1/2 < |\lambda| < 2.$$

By direct calculations we obtain that

$$B^n u(k) = \begin{cases} 2^n u(k+n), & k \geq 0; \\ 2^{n+2k} u(k+n), & -n \leq k < 0; \\ \frac{1}{2^n} u(k+n) & k < -n; \end{cases}$$

and

$$\|B^n u\|^2 \leq \frac{1}{2^{n/4}} \sum_{j \leq n/4} |u(j)|^2 + 2^n \sum_{j > n/4} |u(j)|^2.$$

Therefore

$$\overline{\lim}_{n\to+\infty} \|B^n u\|^{1/n} \leq \max\{\frac{1}{2^{1/4}},\ 2\gamma^+(u)\},$$

where

$$\gamma^+(u) := \overline{\lim}_{n\to+\infty} [\sum_{k>n/4} |u(k)|^2]^{\frac{1}{2n}}.$$

It follows that $u \in F^+$ if $\gamma^+(u) < \frac{1}{2}$.
Similarly $u \in F^-$ if $\gamma^-(u) < \frac{1}{2}$, where

$$\gamma^-(u) := \overline{\lim}_{n\to+\infty} [\sum_{k<-n/4} |u(k)|^2]^{\frac{1}{2n}}.$$

Here

$$F^+ + F^- = F,$$

these subspaces are not closed and are dense subspaces:

$$\overline{F^+} = F,\ \ \overline{F^-} = F.$$

The intersection $F^+ \bigcap F^- \neq \{0\}$ and moreover it is dense:

$$\overline{F^+ \bigcap F^-} = F.$$

There exist a lot of operators satisfying condition (7). In particular, let us consider operators of the form

$$(Pu)(k) = p(k)u(k),$$

where $p(k) \to 0$ as $k \to +\infty$ and $p(k) \to 1$ as $k \to -\infty$. The condition (7) holds if

$$\lim_{n\to+\infty} [\sup\{|p(k)| : k > n\}]^{1/n} < 1/2 \tag{10}$$

and

$$\lim_{n\to+\infty} [\sup\{|1-p(k)| : k < -n\}]^{1/n} < 1/2. \tag{11}$$

Let us take the following concrete sequences $p(k)$ for which the conditions (10)-(11) hold and demonstrate that the properties of the corresponding operators P can be very different.

a)

$$p(k) = \begin{cases} \frac{1}{3^k} & k \geq 0, \\ 1 + \frac{1}{3^{-k}}, & k < 0; \end{cases}$$

b)

$$p(k) = \begin{cases} 0 & k \geq q^+, \\ p(k) & q^- \leq k < q^+, \\ 1, & k < q^-, \end{cases}$$

where $p(k)$ are arbitrary numbers for $q^- \leq k < q^+$.

c)

$$p(k) = \begin{cases} 0 & k \geq q^+, \\ p(k) & q^- \leq k < q^+, \\ 1, & k < q^-, \end{cases}$$

where $p(k)$ take value 0 or 1 only.

d)

$$p(k) = \begin{cases} 0 & k \geq q, \\ 1, & k < q; \end{cases}$$

At the case a) the images ImP and $Im(I-P)$ are not closed and

$$ImP \bigcap Im(I-P) \neq \{0\}.$$

At the case b) the images ImP and $Im(I-P)$ are closed and

$$ImP \bigcap Im(I-P) \neq \{0\}.$$

In particular, $F = ImP + Im(I-P)$, but the last sum is not direct.

At the case c) operator P is a projection, the images ImP and $Im(I-P)$ are closed and

$$F = ImP \bigoplus Im(I-P).$$

At the case d) in addition to the properties from c) the subspace ImP is invariant with respect to operator B and the subspace $Im(I-P)$ is invariant with respect to operator B^{-1}.

The properties of the projection P at the case d) are the most similar ones of the Riesz projection in the hyperbolic case. Let us clarify a geometric sense of the expression (3) for right-side resolvent in the case d).

Let P be an arbitrary projection in the space F. Then we have decomposition

$$F = ImP \bigoplus Im(I-P)$$

and representation of the operator B in the form of operator matrix:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \tag{12}$$

At the hyperbolic case the subspaces ImP and $Im(I-P)$,where P is the Riesz projection, are invariant with respect to B and B^{-1} and representation (12) has the diagonal form:

$$B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}.$$

Here operators $B_{11} : F^+ \to F^+$ and $B_{22} : F^- \to F^-$ are invertible,

$$B^{-1} = \begin{pmatrix} B_{11}^{-1} & 0 \\ 0 & B_{22}^{-1} \end{pmatrix}$$

and

$$(B-\lambda I)^{-1} = \begin{pmatrix} (B_{11}-\lambda I)^{-1} & 0 \\ 0 & (B_{22}-\lambda I)^{-1} \end{pmatrix} = \begin{pmatrix} (B_{11}-\lambda I)^{-1} & 0 \\ 0 & -\frac{1}{\lambda}B_{22}^{-1}(B_{22}^{-1}-\frac{1}{\lambda}I)^{-1} \end{pmatrix}.$$

If subspace ImP is invariant with respect to B (but not with respect to B^{-1}) the representation (12) has upper- triangular form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}.$$

It is known effect for operator matrix in infinity-dimensional case: the inverse to upper-triangular operator matrix B can be not upper- triangular [6, problems 56-57]. From the condition that the space $Im(I-P)$ is invariant with respect to operator B^{-1} we obtain that the inverse operator B^{-1} has lower-triangular form

$$B^{-1} = \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}.$$

It follows from condition (7) that

$$r(B_{11}) < 1, \ \ r(D_{22}) < 1$$

and the right-side resolvent, given by expression (3) with projection P under consideration, is

$$R_r(B;\lambda) = \begin{pmatrix} (B_{11}-\lambda I)^{-1} & 0 \\ 0 & -\frac{1}{\lambda}D_{22}(D_{22}-\frac{1}{\lambda}I)^{-1} \end{pmatrix}.$$

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