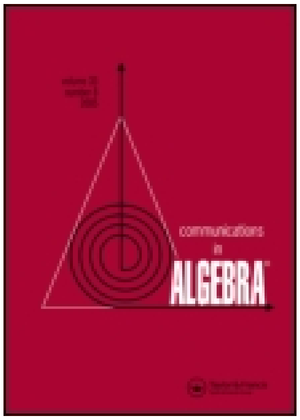

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ON A FINITE GROUP HAVING A NORMAL SERIES WHOSE FACTORS HAVE BICYCLIC SYLOW SUBGROUPS

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We consider the structure of a finite group having a normal series whose factors have bicyclic Sylow subgroups. In particular, we investigate groups of odd order and A_4-free groups with this property. Exact estimations of the derived length and nilpotent length of such groups are obtained.

Key Words: Bicyclic Sylow subgroups; Derived length; A_4-Free groups; Normal series; Nilpotent length.

2000 Mathematics Subject Classification: 20D10.

1. INTRODUCTION

All groups considered in this article will be finite.

By the Zassenhaus Theorem (see Huppert [7], IV, 2.11) the derived subgroup of a group with cyclic Sylow subgroups is a cyclic Hall subgroup such that the corresponding quotient group is also cyclic. Hence the derived length of such group is at most 2.

Recall that a group is bicyclic if it is the product of two cyclic subgroups. The invariants of the groups with bicyclic Sylow subgroups were found in Monakhov and Gribovskaya [9]. In particular, it is proved that the derived length of such groups is at most 6 and the nilpotent length of such groups is at most 4.

Let the group G have a normal series in which every Sylow subgroup of its factors is cyclic. Then G is supersolvable by the Zassenhaus Theorem.

In this article we study groups having a normal series whose factors have bicyclic Sylow subgroups. We prove the following theorem.

Theorem 1.1. *Let G be a solvable group having a normal series such that every Sylow subgroup of its factors is bicyclic. Then the following statements hold*:

(1) *the nilpotent length of G is at most 4 and the derived length of $G/\Phi(G)$ is at most 5*;
(2) *G contains a normal subgroup N such that G/N is supersolvable and N possesses an ordered Sylow tower of supersolvable type*;

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(3) $l_2(G) \leq 2$, $l_3(G) \leq 2$ *and* $l_p(G) \leq 1$ *for every prime* $p > 3$;
(4) *G contains a normal Hall* $\{2, 3, 7\}'$*-subgroup H and H possesses an ordered Sylow tower of supersolvable type.*

Here $\Phi(G)$ *is the Frattini subgroup of G and* $l_p(G)$ *is the p-length of G. A group G is* A_4*-free if there is no section isomorphic to the alternating group* A_4 *of degree* 4.

Corollary 1.2. *Let G be a solvable group having a normal series such that every Sylow subgroup of its factors is bicyclic. If G is an* A_4*-free group then the following statements hold*:

(1) $l_p(G) \leq 1$ *for every prime p*;
(2) *the derived length of* $G/\Phi(G)$ *is at most* 3.

Corollary 1.3. *Let G be a group of odd order having a normal series such that every Sylow subgroup of its factors is bicyclic. Then the following statements hold*:

(1) *G possesses an ordered Sylow tower of supersolvable type*;
(2) *The derived subgroup of G is nilpotent. In particular,* $G/\Phi(G)$ *is metabelian.*

Examples that show accuracy of the estimations in Theorem 1.1 *and Corollary* 1.2 *are constructed*; *see Examples* 3.1–3.3.

2. PRELIMINARIES

In this section, we give some definitions and basic results which are essential in the sequel.

A normal series of a group G is a finite sequence of normal subgroups G_i such that

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_m = G. \quad (1)$$

We call the groups G_{i+1}/G_i the factors of the normal series (1).

Let A be a subgroup of a group G. Then A_G denotes the maximal normal subgroup of G contained in A. Let G be a group of order $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_1 > p_2 > \cdots > p_k$. We say that G has an ordered Sylow tower of supersolvable type if there exists a series

$$1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_{k-1} \leq G_k = G$$

of normal subgroups of G such that for each $i = 1, 2, \ldots, k$, G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G. By $G = [A]B$ we denote the semidirect product with normal subgroup A of G, Z_n is a cyclic group of order n. We use $d(G)$ to denote the derived length of a solvable group G.

Let $\mathfrak{F}$ and $\mathfrak{H}$ be nonempty formations. If G is a group, then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual of G, that is, the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$. We define $\mathfrak{F} \circ \mathfrak{H} = \{G \mid G^{\mathfrak{H}} \in \mathfrak{F}\}$ and call $\mathfrak{F} \circ \mathfrak{H}$ the formation product of $\mathfrak{F}$ and $\mathfrak{H}$ (see Doerk and Hawkes, [3], IV, 1.7). As usually, $\mathfrak{F}^2 = \mathfrak{F} \circ \mathfrak{F}$ and $\mathfrak{F}^n = \mathfrak{F}^{n-1} \circ \mathfrak{F}$ for every natural $n \geq 3$. A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. In this article, $\mathfrak{N}$ and $\mathfrak{A}$ denotes the formations of all nilpotent and all Abelian groups, respectively. The other definitions and terminology about formations could be referred to Doerk and Hawkes [3], Huppert [7], and Shemetkov [11].

Lemma 2.1. *Let G be a bicyclic p-group.*

1. *Let N be a complemented normal subgroup in G. Then*:

   (1.1) *if $p = 2$, then $|N/\Phi(N)| \leq 4$;*
   (1.2) *if $p > 2$, then either $N = G$ or N is cyclic.*

2. *If $p > 2$, then G is metacyclic.*
3. *If $p = 2$, then any normal subgroup of G is generated by at most three elements.*

Proof. 1. It follows from Monakhov and Gribovskaya ([9], Lemma 1).

2. It follows from Huppert ([7], III, 11.5).

3. Let $G = \langle a \rangle \langle b \rangle$ be a bicyclic 2-subgroup and N a normal subgroup of G. Apply induction on $|G| + |G/N|$. First we show that $|N/\Phi(N)| \leq 8$. Assume that $\Phi(N) \neq 1$. Then $\Phi(N)$ is normal in G and by induction, $N/\Phi(N)$ is generated by at most three elements. Hence $|N/\Phi(N)| \leq 8$ and by Huppert ([7], III, 3.15), N is generated by at most three elements. Consequently, $\Phi(N) = 1$ and N is an elementary Abelian group. By the inductive assumption, N is not contained in the proper bicyclic subgroups of G. If $\langle a \rangle N \neq G$, then $\langle a \rangle N = \langle a \rangle (\langle a \rangle N \cap \langle b \rangle)$ is bicyclic, a contradiction. Hence $\langle a \rangle N = G$. Let $T = \langle a \rangle \cap N$. Then $|T| \leq 2$ and G/T is bicyclic 2-subgroup with complemented normal subgroup N/T. By 1.1), $|N/T| \leq 4$. Hence $|N| \leq 8$. The lemma is proved. □

Example 2.2. The calculations in the computer system GAP (see GAP, [4]) show that the group G of order $189 = 3^3 7$ having number 7 in the library SmallGroups,

$$G = \langle a, b, c, d \mid b^3 = c^3 = d^7 = 1, a^3 = c, [a, b] = c^{-1}, \\ [a, d] = d^{-1}, [a, c] = [b, c] = [b, d] = [c, d] = 1 \rangle,$$

is the product of two cyclic subgroups $A = \langle bd \rangle$ of order 21 and $B = \langle ab \rangle$ of order 9. Hence, G is a bicyclic nonprimary group of odd order. There are only three nontrivial cyclic normal subgroups in G: $N_1 = \langle c \rangle$ of order 3, $N_2 = \langle d \rangle$ of order 7, and $N_3 = \langle cd \rangle$ of order 21. Since G/N_i is noncyclic, it follows that G is non-metacyclic. Therefore, the statement of Proposition 2 (Lemma 2.1) is not true for nonprimary groups.

Example 2.3. The bicyclic 2-group G of order 32,

$$G = \langle a, b, c \mid a^2 = b^8 = c^2 = 1, [a, b] = c, [b, c] = b^4, [a, c] = 1 \rangle,$$

(see Huppert [6]), contains a normal elementary Abelian subgroup $N = \langle a \rangle \times \langle b^4 \rangle \times \langle c \rangle$ of order 8 with cyclic group G/N of order 4. This example shows that the estimation of the number of generators in Proposition 3 (Lemma 2.1) is exact.

Recall that $r_p(G)$ is the chief p-rank of the solvable group G (see Huppert [7], VI, 5.2). The chief rank is the maximum of $r_p(G)$ for all $p \in \pi(G)$.

Lemma 2.4. *Let G be a solvable group having a normal series such that every Sylow subgroup of its factors is bicyclic. Then the orders of chief factors of G are p, q^2, or 8, where p and q are primes from $\pi(G)$.*

Proof. Let (1) be a normal series of G such that every Sylow subgroup of its factors is bicyclic. We refine this series to a chief series of G. Let $\overline{N} = N/G_i$ be a minimal normal subgroup of $\overline{G} = G/G_i$ such that $\overline{N} \subseteq \overline{G_{i+1}} = G_{i+1}/G_i$. Since $\overline{G}$ is solvable, $\overline{N}$ is an elementary Abelian p-subgroup for some prime $p \in \pi(G)$. Besides, $\overline{N}$ is normal in a bicyclic Sylow p-subgroup of $\overline{G_{i+1}}$. If $p > 2$, then $\overline{G_{i+1}}$ is metacyclic by Proposition 2 (Lemma 2.1). Hence $|\overline{N}| = p$ or $|\overline{N}| = p^2$. If $p = 2$, then $|\overline{N}| = 2$, 4, or 8 by Proposition 3 (Lemma 2.1). As a result we obtain a chief series with factors of orders p, q^2, or 8. By the Jordan–Hölder Theorem, all chief series of some group are isomorphic. Hence, $r_p(G) \leq 2$ for any prime $p > 2$ and $r_2(G) \leq 3$ by definition of the chief p-rank $r_p(G)$. The lemma is proved. □

Lemma 2.5. *Let G be a group of odd order. Then G has a normal series such that every Sylow subgroup of its factors is bicyclic if and only if the chief rank of G is at most 2.*

Proof. Let G has a normal series such that every Sylow subgroup of its factors is bicyclic. Then the chief rank of G is at most 2 by Lemma 2.4. Conversely, if the chief rank of G is at most 2, then G has a chief series in which every factor either has prime order or is an elementary Abelian of order p^2 for some prime p. The lemma is proved. □

Lemma 2.6. *Let G be a solvable group having a normal series such that every Sylow subgroup of its factors is bicyclic. If M is a maximal subgroup of G, then $|G : M|$ is either a prime or the square of a prime or* 8.

Proof. By Lemma 2.4, G has a chief series

$$1 = G_0 < G_1 < \cdots < G_i < G_{i+1} < \cdots < G_m = G$$

with factors of orders p, q^2, or 8, where p and q are primes. Let $G_i \subseteq M$, but $G_{i+1} \not\subseteq M$. Since M is maximal in G, it follows that $G_{i+1}M = G$ and $|G : M| = |G_{i+1} : G_{i+1} \cap M|$. Because $G_i \subseteq G_{i+1} \cap M$, we have

$$|G_{i+1} : G_{i+1} \cap M| = \frac{|G_{i+1} : G_i|}{|G_{i+1} \cap M : G_i|}$$

and $|G : M|$ is either a prime or the square of a prime or 8. The lemma is proved. □

Lemma 2.7 (Bloom [1], Theorem 3.4). *Let G be a subgroup of $GL(2, q)$ and $q = p^\alpha$, where p is prime. Then, up to conjugacy in $GL(2, q)$, one of the following occurs:*

(1) *G is cyclic;*

(2) $G = QM$, *where* Q *is a subgroup of the* p*-group* $\left\{\left(\begin{smallmatrix}1 & 0\\ \tau & 1\end{smallmatrix}\right) \mid \tau \in GF(q)\right\}$ *and* $M \subseteq N_G(Q)$ *is a subgroup of the group* D *of all diagonal matrices*;
(3) $G = \{Z_u, S\}$, *where* u *divides* $q^2 - 1$, $S : Y \to Y^q$, *for all* $Y \in Z_u$, *and* S^2 *is a scalar* 2*-element in* Z_u;
(4) $G = \{M, S\}$, *where* $M \subseteq D$ *and* $|G : M| = 2$;
(5) $G = \langle SL(2, p^\beta), V\rangle$ (*"Case* 1*") or*

$$G = \left\langle SL(2, p^\beta), V, \begin{pmatrix} b & 0 \\ 0 & \epsilon b\end{pmatrix}\right\rangle,$$

(*"Case* 2*"*), *where* V *is a scalar matrix,* ϵ *generates* $(GF(p^\beta))^*$, $p^\beta > 3$, $\beta|\alpha$. *In Case* 2, $|G : \langle SL(2, p^\beta), V\rangle| = 2$;
(6) $G/\{-I\}$ *is isomorphic to* $S_4 \times Z_u$, $A_4 \times Z_u$ *or* $A_5 \times Z_u$, *if* $p \neq 5$, *where* Z_u *is a scalar subgroup of* $GL(2, q)/\{-I\}$;
(7) G *is not of type* (6), *but* $G/\{-I\}$ *contains* $A_4 \times Z_u$ *as a subgroup of index* 2, *and* A_4 *as a subgroup with cyclic quotient group,* Z_u *is as in type* (6) *with* u *even.*

Lemma 2.8. *Let* H *be an* A_4*-free* p'*-subgroup of* $GL(2, p)$, *where* p *is prime. Then* H *is metabelian.*

Proof. We shall use the result of Lemma 2.7. A subgroup H from Proposition 1 is Abelian. The order of a subgroup H from Proposition 2 is divisible by a prime p. Since the group of all diagonal matrices is Abelian, it follows that a subgroup H from Proposition 3-4 is metabelian. A subgroup H from Proposition 5-7 is not A_4-free. Hence if H is an A_4-free p'-subgroup $GL(2, p)$, then H is metabelian. The lemma is proved. □

Lemma 2.9. *Let* H *be a subgroup of* $GL(3, 2)$. *Then* $H \in \{1, GL(3, 2), Z_2, Z_3, Z_7, Z_2 \times Z_2, Z_4, D_8, S_3, A_4, S_4, [Z_7]Z_3\}$.

Proof. By Huppert ([7], II, 6.14), $GL(3, 2) \simeq PSL(2, 7)$. In view of Huppert ([7], II, 8.27), we conclude that H satisfies the hypotheses of our lemma. □

Lemma 2.10. *Let* G *be a solvable group such that the index of each of its maximal subgroup is either a prime or the square of a prime or* 8. *Then the following statements hold*:

(1) $G \in \mathfrak{N}_{2'} \circ \mathfrak{N}_2 \circ \mathfrak{U}$. *In particular, the nilpotent length of* G *is at most* 4;
(2) G *contains a normal subgroup* N *such that* G/N *is supersolvable and* N *possesses an ordered Sylow tower of supersolvable type*;
(3) $l_2(G) \leq 2$, $l_3(G) \leq 2$ *and* $l_p(G) \leq 1$ *for every prime* $p > 3$. *If* G *is a group of odd order, then* $l_p(G) \leq 1$ *for every prime* $p \in \pi(G)$;
(4) G *contains a normal Hall* $\{2, 3, 7\}'$*-subgroup* H *and* H *possesses an ordered Sylow tower of supersolvable type*;
(5) *If* G *is a group of odd order, then* G *possesses an ordered Sylow tower of supersolvable type.*

Proof. 1. It follows from Gribovskaya ([5], Theorem 2, Corollary 3).

2. By 1) $G \in \mathfrak{N}_{2'} \circ \mathfrak{N}_2 \circ \mathfrak{U}$, i.e., $G^{\mathfrak{U}} \in \mathfrak{N}_{2'} \circ \mathfrak{N}_2$. Hence $G^{\mathfrak{U}} = [T]H$, where T is a $2'$–Hall subgroup, H is a Sylow 2-subgroup. Since $T \in \mathfrak{N}_{2'}$, it follows that T is nilpotent and $G^{\mathfrak{U}}$ possesses an ordered Sylow tower of supersolvable type.

3. We use induction on $|G|$. Let p be a prime divisor of $|G|$. By Huppert ([7], VI, 6.9), we may assume that $O_{p'}(G) = \Phi(G) = 1$ and $G = [F]M$, where the Fitting subgroup $F = F(G) = C_G(F)$ is the unique minimal normal p-subgroup and M is a maximal subgroup of G. Hence, a Sylow p-subgroup $G_p = [F](G_p \cap M) = [F]M_p$, where M_p is a Sylow p-subgroup of M. If $M_p = 1$, then $F = G_p$ and $l_p(G) \leq 1$. Let $M_p \neq 1$. Since $|F| = |G : M|$, it follows that $|F|$ is equal either to p or p^2, or 8. If $|F| = p$, then G/F is a cyclic group whose order divides $(p-1)$. Hence $G_p = F$, a contradiction.

Let $|F| = p^2$. Then G/F is isomorphic to a subgroup of $GL(2, p)$. Since $|GL(2, p)| = (p^2 - p)(p^2 - 1)$, the order of G_p is equal to p^3 and by Huppert ([7], VI, 6.6), $l_p(G) \leq 2$. Since $F = C_G(F)$, G_p is non-Abelian and by Huppert ([7], I, 14.10), it is isomorphic either to a metacyclic group $M_3(p) = \langle a, b \mid a^{p^2} = b^p = 1, a^b = a^{1+p} \rangle = [\langle a \rangle]\langle b \rangle$, or to a group of exponent p. Since $\Omega_1(M_3(p))$ is an elementary Abelian p-subgroup of order p^2, it does not have a complement in $M_3(p)$. Hence G_p is a group of exponent p. If G has odd order or p is not a Fermat prime, then by Huppert and Blackburn ([8], IX, 4.8), $l_p(G) \leq 1$. But now by Huppert and Blackburn ([8], IX, 5.5(b)), $l_p(G) \leq 1$ for $p > 3$.

Finally, let $|F| = 8$. Then $p = 2$ and G/F is isomorphic to a subgroup H of $GL(3, 2)$. In this case, $O_2(G/F) = 1$ and by Lemma 2.9, $H \in \{Z_3, Z_7, S_3, [Z_7]Z_3\}$. Evidently, $l_2(G) \leq 2$.

4. We show that G has a normal Hall π–subgroup G_π for $\pi = \pi(G) \setminus \{2, 3, 7\}$. Since the class of all π-closed subgroups is a saturated formation, by induction we can assume that $O_\pi(G) = 1$ and the Fitting subgroup F is an elementary Abelian p-subgroup whose order divides 2^3, 3^2 or 7^2. Hence the group G/F is isomorphic to a subgroup of $GL(n, p)$ for $p = 2$ and $n \leq 3$, or for $p \in \{3, 7\}$ and $n \leq 2$. Since $\pi(GL(n, p)) \subseteq \{2, 3, 7\}$ for given n and p, it follows that G is a π'–subgroup.

By Monakhov et al. ([10], Corollary 2.4), G_π possesses an ordered Sylow tower of supersolvable type.

5. It follows from Monakhov et al. ([10], Corollary 2.3). □

3. PROOFS OF THEOREM 1.1 AND COROLLARIES 1.2 AND 1.3

Proof of Theorem 1.1

By Lemma 2.6 and Lemma 2.10 (1–4), we must only prove that the derived length of $G/\Phi(G)$ is at most 5.

We first show that $G \in \mathfrak{N} \circ \mathfrak{A}^4$. Apply induction on $|G|$. Assume that $\Phi(G) \neq 1$. Since any quotient group satisfies the hypothesis of the theorem, $G/\Phi(G) \in \mathfrak{N} \circ \mathfrak{A}^4$ by induction. Since $\mathfrak{N} \circ \mathfrak{A}^4$ is a saturated formation, it follows that $G \in \mathfrak{N} \circ \mathfrak{A}^4$. Next we assume that $\Phi(G) = 1$.

Now suppose that the Fitting subgroup $F(G)$ is not a minimal normal subgroup in G. Then $F(G)$ is the direct product of minimal normal subgroups of G, i.e., $F(G) = F_1 \times F_2 \times \cdots \times F_n$, where F_i is a minimal normal subgroup of G for any i and $n \geq 2$. By the inductive assumption, we have $G/F_i \in \mathfrak{N} \circ \mathfrak{A}^4$. Consequently, $G \in \mathfrak{N} \circ \mathfrak{A}^4$, because $\mathfrak{N} \circ \mathfrak{A}^4$ is a formation.

Next we assume that $F = F(G)$ is the unique minimal normal subgroup of G. Besides, $F = C_G(F)$ and $G = [F]M$, where M is a maximal subgroup of G. Since $|F| = |G : M|$, it follows by Lemma 2.6 that $|F|$ is equal to p, p^2 or 8, where p is prime.

If $|F| = p$, then G/F is cyclic, since it is the subgroup of $\mathrm{Aut}F = Z_{p-1}$. Hence $G/F \in \mathfrak{A}$. Let $|F| = p^2$. Then G/F is isomorphic to an irreducible solvable subgroup of $GL(2, p)$. By Monakhov and Gribovskaya ([9], Lemma 3), $G/F \in \mathfrak{A}^4$.

It remains to study the case $|F| = 8$. Then G/F is isomorphic to a solvable subgroup H of $GL(3, 2)$. Let's notice that F is the maximal normal 2-subgroup of G, i.e., $F = O_2(G)$. Hence $O_2(G/F) = 1$. By Lemma 2.9, $G/F \in \{Z_3, S_3, Z_7, [Z_7]Z_3\}$ and $G/F \in \mathfrak{A}^2 \subseteq \mathfrak{A}^4$.

From all the above, we proved that $G/F \in \mathfrak{A}^4$. As F is nilpotent, $G \in \mathfrak{N} \circ \mathfrak{A}^4$. Since $F/\Phi(G)$ is Abelian and $(G/\Phi(G))/(F/\Phi(G)) \simeq G/F$, it follows that $G/\Phi(G) \in \mathfrak{A}^5$ and $d(G/\Phi(G)) \leq 5$. The theorem is proved.

Proof of Corollary 1.2

1. By Proposition 3 (Theorem 1.1), we obtain $l_2(G) \leq 2$, $l_3(G) \leq 2$, and $l_p(G) \leq 1$ for every prime $p > 3$. Now we show that $l_p(G) \leq 1$, where $p \in \{2, 3\}$. By Huppert ([7], VI, 6.9), we may say that $O_{p'}(G) = \Phi(G) = 1$. By Lemma 2.4, the Fitting subgroup $F = F(G)$ is the unique minimal normal subgroup of order p^α, where $\alpha \leq 3$ for $p = 2$ and $\alpha \leq 2$ for $p = 3$. In particular, $C_G(F) = F$ and $G = [F]M$ for some maximal subgroup M of G. If $|F| = p$, then G/F is isomorphic to a subgroup of order $p - 1$ and $l_p(G) \leq 1$. If $|F| = 4$, then $\mathrm{Aut}(F(G)) \simeq GL(2, 2) \simeq S_3$. Hence either $G/F(G) \simeq Z_3$ or $G/F(G) \simeq S_3$. If $G/F(G) \simeq Z_3$, then $G \simeq A_4$. If $G/F(G) \simeq S_3$, then $G \simeq S_4$. It means that G is not A_4-free, a contradiction.
   Now let $|F| = 8$. Then G/F is isomorphic to a subgroup of $GL(3, 2)$. Since $O_2(G/F) = 1$, it follows by Lemma 2.9, that $G/F \in \{Z_3, S_3, Z_7, [Z_7]Z_3\}$. In all cases, except $G/F \simeq S_3$, we have $l_2(G) \leq 1$. Suppose that G/F is isomorphic to S_3. We may construct the subgroup $H = [F]Z_3$ in G. Then the alternating group A_4 of degree 4 is contained in H, a contradiction.
   Let $|F| = 9$. Then G/F is isomorphic to a subgroup of $GL(2, 3)$ and $O_3(G/F) = 1$. It is well known that $H \in \{1, Z_2, Z_4, Z_8, Z_2 \times Z_2, D_8, Q_8, SD_{16}, SL(2, 3), GL(2, 3)\}$. In any case, except $G/F \cong SL(2, 3)$ and $G/F \cong GL(2, 3)$, F is a Sylow 3-subgroup in G and $l_3(G) \leq 1$. Since $SL(2, 3)$ and $GL(2, 3)$ are not A_4-free, we have a contradiction.
2. We use induction on $|G|$. We first prove that $G \in \mathfrak{N} \circ \mathfrak{A}^2$. By induction, we can assume that $\Phi(G) = 1$ and G has the unique minimal normal subgroup which coincides with Fitting subgroup $F = F(G)$. By Proposition 1 (Corollary 1.2), $l_p(G) \leq 1$. Hence F is a Sylow p-subgroup of G. Besides, $F = C_G(F)$ and F has a complement M in G, where M is a maximal subgroup of G. By Lemma 2.6, $|F|$ is equal to p, p^2 or 8, where p is prime.
   If $|F| = p$, then G/F is cyclic, since it is the subgroup of $\mathrm{Aut}F = Z_{p-1}$. Hence G/F is Abelian. Let $|F| = p^2$. Then G/F is isomorphic to an irreducible solvable p'-subgroup H of $GL(2, p)$. By Lemma 2.8, H is metabelian, i.e. $G/F \in \mathfrak{A}^2$.

Now let $|F| = 8$. Then G/F is isomorphic to a subgroup of $GL(3, 2)$. By Lemma 2.9, $G/F \in \{Z_3, Z_7, [Z_7]Z_3\}$. Then H is metabelian and $G/F \in \mathfrak{A}^2$.
So, in any case $G/F \in \mathfrak{A}^2$. Since $F/\Phi(G)$ is Abelian and $(G/\Phi(G))/(F/\Phi(G)) \simeq G/F$, it follows that $G/\Phi(G) \in \mathfrak{A}^3$ and $d(G/\Phi(G)) \leq 3$. The corollary is proved.

Proof of Corollary 1.3

1. By Lemma 2.10 (5), our assertion holds.
2. We show that the derived subgroup of G is nilpotent. We use induction on $|G|$. Without loss of generality, we may assume that $\Phi(G) = 1$ and G has a unique minimal normal subgroup which coincides with Fitting subgroup $F = F(G)$. Then F is an elementary Abelian p-subgroup for some prime p. Since $\Phi(G) = 1$, it follows that G has a maximal subgroup M such that $G = [F]M$. Because $|F| = |G : M|$, we have by Lemma 2.6, that $|F|$ is equal to p or p^2. By Proposition 3 (Lemma 2.10), $l_p(G) = 1$. Hence F is a Sylow p-subgroup of G and G/F is a p'-subgroup. In the solvable groups the Fitting subgroup coincides with its centralizer in G, and hence G/F is isomorphic to a subgroup of Aut$F$.

If $|F| = p$, then G/F is cyclic and $G' \subseteq F$. Let $|F| = p^2$. Then G/F is isomorphic to an irreducible solvable p'-subgroup H of $GL(2, p)$. By Dixon ([2], Theorem 5.2), H is Abelian and $G' \subseteq F$. So, in any case, the derived subgroup of G is nilpotent.

Since $F/\Phi(G)$ is Abelian, it follows that $G/\Phi(G)$ is metabelian. The corollary is proved.

Example 3.1. Let E_{7^2} be an elementary Abelian group of order 7^2. The automorphism group of E_{7^2} is the general linear group $GL(2, 7)$ with cyclic center $Z = Z(GL(2, 7))$ of order 6. We choose a subgroup C of order 2 in Z. Evidently, C is normal in $GL(2, 7)$. The calculations in the computer system GAP show that $GL(2, 7)$ has a subgroup S of order 48 such that S/C is isomorphic to the symmetric group S_4 of degree 4. The semidirect product $G = [E_{7^2}]S$ is a group of order $2352 = 2^4 7^2 3$. In particular, $\Phi(G) = 1$. The nilpotent length of G is equal to 4, the derived length of G is equal to 5. The group G has the chief series

$$1 \subset E_{7^2} \subset [E_{7^2}]Z_2 \subset [E_{7^2}]Q_8 \subset [[E_{7^2}]Q_8]Z_3 \subset [E_{7^2}]S = G$$

with bicyclic factors

$$E_{7^2}, ([E_{7^2}]Z_2)/(E_{7^2}) \simeq Z_2, \qquad ([E_{7^2}]Q_8)/([E_{7^2}]Z_2) \simeq E_4,$$
$$([[E_{7^2}]Q_8]Z_3)/([E_{7^2}]Q_8) \simeq Z_3, \qquad (G/[[E_{7^2}]Q_8]Z_3) \simeq Z_2.$$

Hence, the estimations of the nilpotent length and the derived length, which are obtained in Theorem 1.1, are exact.

Example 3.2. Let E_{5^2} be an elementary Abelian group of order 5^2. The automorphism group of E_{5^2} is the general linear group $GL(2, 5)$. The group $GL(2, 5)$ has a subgroup, which is isomorphic to the symmetric group S_3 of degree 3.

The semidirect product $G = [E_{5^2}]S_3$ is an A_4-free group with identity Frattini subgroup. The derived length of G is equal to 3. The group G has the chief series

$$1 \subset E_{5^2} \subset [E_{5^2}]Z_3 \subset [E_{5^2}]S_3 = G$$

with bicyclic factors

$$E_{5^2}, \qquad ([E_{5^2}]Z_3)/(E_{5^2}) \simeq Z_3, \qquad ([E_{5^2}]S_3)/([E_{5^2}]Z_3) \simeq Z_2.$$

Consequently, the estimation of the derived length, which is obtained in Corollary 1.2, is exact.

Example 3.3. It is well known that S_4 has the normal series

$$1 \leq E_4 \leq A_4 \leq S_4$$

with bicyclic factors and $l_2(S_4) = 2$. The group $G = [E_{3^2}]SL(2,3)$ has the normal series

$$1 \leq E_{3^2} \leq [E_{3^2}]Z_2 \leq [E_{3^2}]Q_8 \leq [E_{3^2}]SL(2,3)$$

with bicyclic factors and $l_3(G) = 2$.

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