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Injective Objects in the Category of Stratified Spaces

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Abstract—The paper deals with the category of stratified spaces. For this category we establish the existence of the absolute extensors and study their connection with the absolute extensors in the category of filtered spaces. To this end, we introduce and investigate the absolute extensors reflecting the properties of both the filtered category and the stratified category. We determine the contractibility property as well as the local contractibility property for the stratified absolute extensors. It is shown that the family of strata for a stratified absolute neighbourhood extensor is equicontinuously locally extendable.

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Introduction. Spaces with countable filtrations consisting of closed subsets are of great interest in modern topology. One of the fruitful approaches to the research in those spaces is the study of extensor properties in the category of filtered spaces (or the $\mathcal{N}$-category). The objects in this category are the filtered metric spaces, and its morphisms are the filtered maps, i.e., the maps preserving filtrations. Arising from that approach, the theory of absolute extensors and retracts in the filtered category develops and successfully supplements Milnor's classical results about the cell complexes and the properly filtered spaces ([1], P. 255; [2], P. 405). Fundamentals of this theory are contained in the work [3].

However, it is impossible in principal to track topological and homotopic properties of strata in the filtered spaces within the framework of the filtered category (although it is very desirable in many situations). It is connected with the fact that the filtered maps retaining the elements of a filtration as invariants do not preserve the strata. It follows from this fact, for example, that any injective object in the $\mathcal{N}$-category (an $\mathcal{N}$-AE-space) is homotopically equivalent to a point. But nevertheless there exist $\mathcal{N}$-AE-spaces with homotopically nontrivial strata. Therefore, there is a need for a finer theory which gives an information about strata and describes a situation adequately.

A way out of the situation is possible if one introduces a new category (the $\mathcal{S}$-category) whose objects are the filtered metric spaces as before, and whose morphisms are the maps preserving the strata (the stratified maps). In that case, the restriction of the homotopical equivalence in this category to any stratum is the homotopical equivalence, and any stratum of an injective object of the $\mathcal{S}$-category is an absolute retract. However, there is no a priori answer to the question whether the collection of such injective objects is large. It is also not clear whether properties of those objects are good enough.

The main result in the paper is the presentation of a sufficient collection of the injective objects in the $\mathcal{S}$-category. In fact, we establish a stronger result (Theorem 2). Namely, we show that not only a stratified map but any filtered map taking its values in an $\mathcal{S}$-AE-space admits a stratified extension. As an immediate consequence of this theorem, we obtain the basic properties of the stratified absolute extensors. In particular, we establish the connection between the $\mathcal{N}$-AE-spaces and the $\mathcal{S}$-AE-spaces (Theorem 4). We also study other properties of the $\mathcal{S}$-AE-spaces.

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As a hypothesis in the category of spaces with countable filtrations, we will formulate a generalization of Torunczyk's well-known theorem on a characterization of the Hilbert cube $Q = [-1;1]^\omega$ [4]. Recall that this theorem states the following: *Any compact metric AE-space which is universal with respect to the class of metric compacta is homeomorphic to Q.* To this end, we introduce an analog of the Hilbert cube in the category of the filtered spaces. Since Q is homeomorphic to its own countable power Q^ω, we may consider the filtration on Q defined by putting $\deg \overline{q} = \max\{i \mid q_i \neq \overline{0}\}$, where $\overline{q} = (q_1, q_2, \dots) \in Q^\omega$ and $\overline{0} = (0, 0, \dots) \in Q = [-1;1]^\omega$. We also set $\deg \overline{0} = 1$. It follows from Theorem 2 that $Q \in \mathcal{S}$-AE. Moreover, it can be shown that Q is an $\overline{\mathcal{N}}$-universal space in the following sense: For every $\overline{\mathcal{N}}$-map $f : Z \to Q$ from a compact metric $\overline{\mathcal{N}}$-space Z and for every $\varepsilon > 0$ there exists an $\mathcal{S}$-embedding $\widetilde{f} : Z \to Q$ such that $\operatorname{dist}(f, \widetilde{f}) < \varepsilon$, where $\operatorname{dist}(f, g)$ is the distance between functions f and g. Our hypothesis is the following: *Any compact metric $\overline{\mathcal{N}}$-space $X \in \mathcal{S}$-AE which is an $\overline{\mathcal{N}}$-universal space is $\overline{\mathcal{N}}$-homeomorphic to Q.* It is very plausible but complicated and requires a deep research in geometry of a countable filtration on Q. Notice that the hypothesis is valid in the case of a finite filtration.

1. The main notions and results. *A generalized filtered space* (or an *$\overline{\mathcal{N}}$-space*) is a metric space X with a chosen sequence of closed subsets $X_1 \subset X_2 \subset \dots \subset X_\infty = X$. This sequence is called *a filtration*. In this case the union $\cup\{X_i \mid i < \infty\}$ is an F_σ-set in X and does not coincide with X, in general. The subset $X_i \subset X$ is called *the ith element of the filtration* $\{X_i\}$ in the space X. The subsets $X^i \rightleftharpoons X_i \setminus X_{i-1} \subset X$, $i < \infty$, and $X^\infty = X \setminus \bigcup\limits_{i<\infty} X_i$ are called the *i-stratum* and the *∞-stratum* of the filtration $\{X_i\}$, respectively.

For defining a filtration in a space X, it suffices to associate to each point $x \in X$ its *degree* $\deg x \in \mathbb{N}$ in such a way that for every $i < \infty$ the ith element $X_i \rightleftharpoons \{x \in X \mid \deg x \leqslant i\}$ of the filtration is closed in X.

The degree of an arbitrary *set* $M \subset X$, denoted by $\deg M$, is defined to be $\min\{i \mid M \cap X^i \neq \emptyset\}$. *The degree* $\deg X$ of an *$\overline{\mathcal{N}}$-space* X is defined to be $\min\{i \mid X_i \neq \emptyset\}$. In this case $\deg X = \infty$ if and only if $X = X_\infty$. Since the spaces with the infinite degree are not interesting, we will always assume that $\deg X < \infty$.

A filtration of a space X is said to be *trivial* if the degree of every point in the space coincides with the degree of X. A metric subspace $A \subset X$ with the induced filtration, i.e., with the filtration formed by the sets $A_i = A \cap X_i$, $i \in \mathbb{N}$, is called an *$\overline{\mathcal{N}}$-subspace* of an $\overline{\mathcal{N}}$-space X.

A linear topological space L with a chosen filtration $\{L_i\}$ is called *a linear topological $\overline{\mathcal{N}}$-space* if each L_i is a linear subspace of L.

A continuous map $f : X \to Y$ between $\overline{\mathcal{N}}$-spaces is said to be

- *filtered* (or an *$\overline{\mathcal{N}}$-map*) if $f(X_n) \subset Y_n$ for all $n \geqslant 1$, i.e., $\deg f(x) \leqslant \deg x$ for all $x \in X$;
- *stratified* (or an *$\mathcal{S}$-map*) if $f(X^n) \subset Y^n$ for all $n \geqslant 1$, i.e., $\deg f(x) = \deg x$ for all $x \in X$.

It follows from the definition that a point $x \in X$ having the infinite degree can be transformed by an $\overline{\mathcal{N}}$-map f onto any point of the space Y, and by an $\mathcal{S}$-map f it can be carried only to a point with the infinite degree.

A filtered space (or an *$\mathcal{N}$-space*) is a metric space X with a filtration $X_1 \subset X_2 \subset \cdots$ such that the equality $\bigcup\limits_{i=1}^{\infty} X_i = X$ holds, i.e., we have $X^\infty = \emptyset$.

It is obvious that the category $\mathcal{N}$ is *a full subcategory* of the category $\overline{\mathcal{N}}$. Let us construct a functor $F : \overline{\mathcal{N}} \to \mathcal{N}$ which is right inverse for the embedding $\mathcal{N} \hookrightarrow \overline{\mathcal{N}}$. To do this we assign to a given $\overline{\mathcal{N}}$-space X with a filtration $X_1 \subset X_2 \subset \dots \subset X_\infty$ the $\mathcal{N}$-space $F(X) \rightleftharpoons \bigcup\limits_{i<\infty} X_i$ with the filtration $X_1 \subset X_2 \subset \cdots$. And, for an $\overline{\mathcal{N}}$-map $f : X \to Y$ between $\overline{\mathcal{N}}$-spaces, we define the map $F(f)$ to be the restriction of f to $F(X)$.

Let us consider now the notions related to the extensions of morphisms in the category of $\overline{\mathcal{N}}$-spaces. Let A be a closed $\overline{\mathcal{N}}$-subspace of an $\overline{\mathcal{N}}$-space X and $f : A \to Y$ be an $\overline{\mathcal{N}}$-map (an $\mathcal{S}$-map). That map is called *a partial* $\overline{\mathcal{N}}$*-map*. If there exists an $\overline{\mathcal{N}}$-map (an $\mathcal{S}$-map) $\widehat{f}: X \to Y$ such that $\widehat{f}\restriction_A = f$, then the map $\widehat{f}$ is said to be an $\overline{\mathcal{N}}$*-extension* (an $\mathcal{S}$*-extension*) of f.

Let Y be a generalized filtered space. We call Y *an absolute* $\overline{\mathcal{N}}$*-extensor* and write $Y \in \overline{\mathcal{N}}$-AE (*an absolute neighborhood* $\overline{\mathcal{N}}$*-extensor* and write $Y \in \overline{\mathcal{N}}$-ANE) if the following property is fulfilled. For any metric $\overline{\mathcal{N}}$-space X and any its closed $\overline{\mathcal{N}}$-subspace A, each partial $\overline{\mathcal{N}}$-map $f: A \to Y$ admits an $\overline{\mathcal{N}}$-extension $\widehat{f}: X \to Y$ (a neighborhood $\overline{\mathcal{N}}$-extension $\widehat{f}: U \to Y$). Analogously one introduces the notion of $\mathcal{S}$-A[N]E-*space* in the category of $\mathcal{S}$-spaces. It is not hard to see that the condition $X \in \mathcal{S}$-A[N]E implies the existence of a point with a finite degree in any neighborhood of an arbitrary point $x \in X$.

Along with the extension problems for $\overline{\mathcal{N}}$-maps and $\mathcal{S}$-maps, we consider it expedient to pose a problem on a combined extension of those maps. An $\overline{\mathcal{N}}$-space X is called *a stratified absolute* $\overline{\mathcal{N}}$*-extensor* (*a stratified absolute neighborhood* $\overline{\mathcal{N}}$*-extensor*) if the following property is fulfilled. For any partial $\overline{\mathcal{N}}$-map $Z \hookleftarrow A \xrightarrow{\varphi} X$ there exists an $\overline{\mathcal{N}}$-extension $\psi: Z \to X$ to the whole space Z which is stratified on $Z \setminus A$ (there exists a neighborhood $\overline{\mathcal{N}}$-extension $\widehat{f}: U \to X$ to a neighborhood U of the set A in Z which is stratified on $U \setminus A$). It is obvious that every stratified $\overline{\mathcal{N}}$-A[N]E-space X is an $\mathcal{S}$-A[N]E-space as well as an $\overline{\mathcal{N}}$-A[N]E-space.

For $\overline{\mathcal{N}}$-spaces we have the following generalization of the Kuratowski—Wojdysławski theorem. *For every* $\overline{\mathcal{N}}$*-space* X *there exist a normed linear* $\overline{\mathcal{N}}$*-space* Z *and an* $\overline{\mathcal{N}}$*-homeomorphism* $h: X \to Y$ *such that* Y *is a closed filtered subset of* Z. Arguments for proving this result are similar to those in [3] (P. 45), where the Kuratowski—Wojdysławski theorem is proved for the $\mathcal{N}$-category.

Further, we will introduce the notions of the product and the sum in the category of $\overline{\mathcal{N}}$-spaces. Let X_k be an $\overline{\mathcal{N}}$-space with a filtration $(X_k)_1 \subset (X_k)_2 \subset \dots \subset (X_k)_\infty$ and a metric d_k for each $k \in \mathbb{N}$. Consider the product $\Pi = \prod\limits_{k\in\mathbb{N}} X_k$ of the topological spaces X_k. We introduce a filtration on Π in the following way. We define the degree of a point $x = \{x_k\} \in \Pi$ to be the number given by $\deg x = \sup\limits_{k\in\mathbb{N}}\{\deg x_k\} \in \overline{\mathbb{N}}$. Since the space Π is metrizable and the set $\Pi_i \rightleftharpoons \{x \in \Pi \mid \deg x \leqslant i\}$ is closed in Π, we conclude that Π is an $\overline{\mathcal{N}}$-space. The latter is called *a topological product of* $\overline{\mathcal{N}}$*-spaces*. It is easy to show that the class of $\overline{\mathcal{N}}$-AE-spaces is closed under the countable product operation of $\overline{\mathcal{N}}$-spaces.

In the case when $\overline{\mathcal{N}}$-spaces X_k are $\mathcal{N}$-spaces, i.e., $(X_k)^\infty = \emptyset$, $k \in \mathbb{N}$, we will consider an $\mathcal{N}$-space $F(\Pi) = \{x \in \Pi \mid \deg x < \infty\}$. This space is called *a topological product of* $\mathcal{N}$*-spaces* X_k. It is not hard to see that the class of $\mathcal{N}$-AE-spaces is closed under the countable product operation of $\mathcal{N}$-spaces.

Assume now that we are given a family of $\overline{\mathcal{N}}$-maps $\{f_k : X \to Y_k\}_{k\in\mathbb{N}}$. Let us consider the diagonal product $f: X \to \prod\limits_{k\in\mathbb{N}} Y_k$ for the maps from this family, i.e., the map defined by $f(x) = \{f_k(x)\}$, $x \in X$. It is not hard to see that f is an $\overline{\mathcal{N}}$-map into the $\overline{\mathcal{N}}$-product $\prod\limits_{k\in\mathbb{N}} Y_k$. The following two corollaries follow from the definition of the degree of a point in the product of $\overline{\mathcal{N}}$-spaces.

Corollary 1. Assume that at least one of maps f_k is stratified. Then the diagonal product f is also stratified.

Corollary 2. Let X_1 be an $\overline{\mathcal{N}}$-A[N]E-space and X_2 be a stratified $\overline{\mathcal{N}}$-A[N]E-space. Then the product $X = X_1 \times X_2$ is a stratified $\overline{\mathcal{N}}$-A[N]E-space.

Let us consider the sum S of topological spaces X_k such that every X_k is an $\overline{\mathcal{N}}$-space. For a point $x \in S$ such that $x \in X_k$, we set that its degree equals the degree of x considered as a point of the $\overline{\mathcal{N}}$-space X_k. Since S is a metrizable space and the ith element of the filtration $S_i \rightleftharpoons \{x \in S \mid \deg x \leqslant i\} = \coprod\limits_{k\in\mathbb{N}} (X_k)_i$ is closed in S, we have the notion of *the topological sum of* $\overline{\mathcal{N}}$*-spaces*.

It is not hard to prove the following statement.

Proposition 1. *Let* $X_1, X_2, X_3, \ldots$ *be a sequence of* $\overline{\mathcal{N}}$*-spaces and* $d \in \overline{\mathbb{N}}$ *be the lower limit* $\underline{\lim} \deg X_k$ *of the sequence* $\{\deg X_k\}$*. Then one can obtain an* $\overline{\mathcal{N}}$*-space* Z *by adjoining to the topological sum* $\coprod_{k\in\mathbb{N}} X_k$ *a single point* s *such that each* X_k *is contained in the* 2^{-k}*-neighborhood* $N(s; 2^{-k}))$ *of the point* s *and* $\deg s \leqslant d$.

The $\overline{\mathcal{N}}$-space $Z = \{s\} \sqcup X_1 \sqcup X_2 \sqcup \cdots$ described in Proposition 1 is called *the one-point complement* of the topological sum of $\overline{\mathcal{N}}$-spaces.

Next we define the notion of the homotopy in the $\overline{\mathcal{N}}$-category. For this purpose, we consider only the trivial filtration on the segment $[0; 1]$ such that the degree of every point equals 1. Then, for an $\overline{\mathcal{N}}$-space X with a filtration $\{X_i\}$, the sets $X_i \times [0; 1]$ are the elements of the filtration in the product $X \times [0; 1]$. Filtered maps ($\mathcal{S}$-maps) $f, g : X \to Y$ are said to be $\overline{\mathcal{N}}$*-homotopic* ($\mathcal{S}$*-homotopic*) if there exists an $\overline{\mathcal{N}}$*-homotopy* (an $\mathcal{S}$*-homotopy*) connecting f with g, i.e., an $\overline{\mathcal{N}}$-map (an $\mathcal{S}$-map) $F : X \times [0; 1] \to Y$ such that $F_0 = f$ and $F_1 = g$.

We will consider the contractibility properties in the $\mathcal{S}$-category. We say that an $\overline{\mathcal{N}}$-space X is $\mathcal{S}$*-contractible* if there exists an $\overline{\mathcal{N}}$-homotopy $H : X \times [0; 1] \to X$ such that

1) $H_t : X \to X$ is an $\mathcal{S}$-map for each $t < 1$;

2) $H_0 = \mathrm{Id}_X$, H_1 maps the space X into a point $x \in X$ (it is clear that x must belong to $X_{\deg X}$).

An $\overline{\mathcal{N}}$-space X is said to be *locally* $\mathcal{S}$*-contractible at a point* $x_0 \in X$ if for any $\varepsilon > 0$ there exist a neighborhood U of the point x_0 and an $\overline{\mathcal{N}}$-homotopy $H : U \times [0; 1] \to X$ such that

1) $H_t : U \to X$ is an $\mathcal{S}$-map for each $t < 1$;

2) $H_0 = \mathrm{Id}_U$, H_1 maps U into a point $x_1 \in X$ with $\deg x_1 < \infty$ (it is clear that x_1 must belong to $X_{\deg U}$);

3) $\operatorname{diam} H(x_0, [0; 1]) \leqslant \varepsilon$.

A generalized filtered space X is said to be *locally* $\mathcal{S}$*-contractible* ($X \in \mathcal{S}$-LC) if it is locally $\mathcal{S}$-contractible at every its point.

2. The existence of stratified absolute extensors. We begin with the following

Definition. A generalized filtered space W is said to be $\mathcal{S}$*-terminal provided that for every* $\overline{\mathcal{N}}$*-space* X *there exists a stratified map*

$$f : X \to W.$$

Theorem 1. *There exists an* $\mathcal{S}$*-terminal space.*

Proof. Let us consider a countable set $\{A_{ij} \mid i, j \in \mathbb{N}\}$ consisting of distinguishable points. It means that any two points A_{ij}, $A_{i'j'}$ of this set coincide if and only if the equality $(i, j) = (i', j')$ holds. For every pair $i, j \in \mathbb{N}$ we consider an $\overline{\mathcal{N}}$-space Con_{ij} which is homeomorphic to the cone over the point A_{ij} with the vertex at 0. Hence, we have $\mathrm{Con}_{ij} = \{t{\cdot}A_{ij} \mid 0 \leqslant t \leqslant 1, 0{\cdot}A_{ij} = 0\}$. We assume that for any points A_{ij} and $A_{i'j'}$ the condition $t{\cdot}A_{ij} = t{\cdot}A_{i'j'}$ holds if and only if either $t = 0$ or $t \neq 0$ and $(i, j) = (i', j')$. Thus, any two distinct cones meet only at the vertex. We set

$$\deg(t{\cdot}A_{ij}) = \begin{cases} 1, & \text{if } t = 0; \\ j, & \text{if } t > 0. \end{cases}$$

We will show that the $\overline{\mathcal{N}}$-product $W \rightleftharpoons \prod\limits_{i=1}^{\infty}\prod\limits_{j=1}^{\infty} \mathrm{Con}_{ij}$ is a sought-for $\mathcal{S}$-terminal space. To this end, we consider an arbitrary metric space X with a bounded metric d (without loss of generality we may assume that the inequality $\operatorname{diam} X \leqslant 1$ holds).

Further we will consider two cases.

1) First, let us study the case when the space X has no isolated points.

It follows from the Stone theorem ([5], P. 416) that X has a σ-discrete base $\{\sigma_i\}_{i=1}^{\infty}$ consisting of the countable union of discrete families σ_i. For every family σ_i and for each j we denote by σ_{ij} the family $\{S_\alpha \in \sigma_i \mid \deg S_\alpha = j\}$. It is clear that the equality $\sigma_i = \bigcup\limits_{j=1}^{\infty} \sigma_{ij}$ holds. We denote by $\cup\sigma_{ij}$ the body $\cup\{S_\alpha \in \sigma_{ij}\}$ of the family σ_{ij}.

Since each σ_{ij} is an open discrete family, we can define the continuous maps $g_{ij} : X \to \mathrm{Con}_{ij}$ for all $i, j = 1, 2, \ldots$ as follows:

$$g_{ij}(x) = \begin{cases} 0, & \text{if } x \notin \cup\sigma_{ij}; \\ \mathrm{d}(x, X \setminus S_\alpha) \cdot A_{ij}, & \text{if } x \in S_\alpha \in \sigma_{ij}, \end{cases}$$

where $\mathrm{d}(x, A)$ is the distance from a point x to a set A. Note that the definition of the filtration for the space Con_{ij} yields the formula

$$\deg g_{ij}(x) = \begin{cases} 1, & \text{if } x \notin \cup\sigma_{ij}; \\ j, & \text{if } x \in S_\alpha \in \sigma_{ij}. \end{cases}$$

Now let us show that g_{ij} is an $\overline{\mathcal{N}}$-map, i.e., the inequality $\deg x \geqslant \deg g_{ij}(x)$ holds for each $x \in X$. Indeed, if $x \in S_\alpha \in \sigma_{ij}$, then we have $\mathrm{d}(x, X \setminus S_\alpha) > 0$ and, consequently, the equality $\deg g_{ij}(x) = j$ holds. Moreover, we have the inequality $\deg x \geqslant \deg S_\alpha$ and, in addition, the equality $\deg S_\alpha = j$. Hence, we obtain the estimate $\deg x \geqslant \deg g_{ij}(x)$. In the case when $x \notin \cup\sigma_{ij}$ we have $\deg g_{ij}(x) = 1$ and $\deg x \geqslant 1$.

Let us check now that the diagonal product $\varphi : X \to W$, $\varphi(x) = \{g_{ij}(x) \mid i, j \in \mathbb{N}\}$, $x \in X$, being an $\overline{\mathcal{N}}$-map, is a desired $\mathcal{S}$-map. For an arbitrary point $x \in X$ with a finite degree $\deg x = j$ there exists a neighborhood U which is an element of the base $\{\sigma_i\}_{i=1}^{\infty}$ such that the equality $\deg U = j$ is true. Assume that the condition $U \in \sigma_{ij}$ holds. Then, considering the map $g_{ij} : X \to \mathrm{Con}_{ij}$, we have $\deg g_{ij}(x) = \deg U = \deg x$. These equalities together with the relations $g_{ij}(x) \neq 0$ and $\deg g_{ij}(x) = j$ for all $x \in \sigma_{ij}$ imply that the value $\deg \varphi(x) = \sup\{\deg g_{ij}(x)\}$ coincides with $j = \deg x$.

Since a point $x_0 \in X$ with the infinite degree is not an isolated point, there exists a neighborhood base $U_1 \in \sigma_{i_1 j_1}, U_2 \in \sigma_{i_2 j_2}, \ldots$ at the point x_0 such that $\{j_k\} \to \infty$. Then we get $\deg \varphi(x_0) = \sup\{\deg g_{ij}(x_0)\} = \lim\limits_{k\to\infty} \deg g_{i_k j_k}(x_0) = \lim\limits_{k\to\infty} j_k = \infty$, i.e., the equality $\deg \varphi(x_0) = \deg x_0$ holds.

2) In the second case when the space X contains isolated points we will consider the product of the $\overline{\mathcal{N}}$-space X and the segment $[0; 1]$. Since the space $X \times [0; 1]$ has no isolated points, there exists an $\mathcal{S}$-map $f : X \times [0; 1] \to W$. Then the restriction $f\restriction_{X\times\{0\}} : X \to W$ is the sought-for $\mathcal{S}$-map. □

In [3] (P. 50) it is shown that the space $F\Big(\prod\limits_{i,j=1}^{\infty} \mathrm{Con}_{ij}\Big)$ is *universal* for $\mathcal{N}$-spaces. This means that for any $\mathcal{N}$-space X there exists an $\mathcal{N}$-subspace $A \subset F\Big(\prod\limits_{i,j=1}^{\infty} \mathrm{Con}_{ij}\Big)$ which is $\mathcal{N}$-homeomorphic to X. In just the same way, it can be shown that the space $\prod\limits_{i,j=1}^{\infty} \mathrm{Con}_{ij}$ is universal for $\overline{\mathcal{N}}$-spaces. It turns out that the space $\prod\limits_{i,j=1}^{\infty} \mathrm{Con}_{ij}$ is closely connected with the stratified extensors.

Theorem 2. *Let an $\overline{\mathcal{N}}$-AE-space X_i be $\mathcal{S}$-terminal for every $i \geqslant 1$. Then the $\overline{\mathcal{N}}$-space $X \rightleftharpoons \prod\limits_{i\geqslant 1} X_i$ is a stratified $\overline{\mathcal{N}}$-AE-space.*

It follows from Theorem 2 that $X \rightleftharpoons \prod\limits_{i\geqslant 1} X_i$ is an $\mathcal{S}$-AE-space.

It is obvious that in Theorem 1 the $\overline{\mathcal{N}}$-space $W = \prod\limits_{i=1}^{\infty}\prod\limits_{j=1}^{\infty} \mathrm{Con}_{ij}$ is $\overline{\mathcal{N}}$-homeomorphic to its own countable power W^{ω}. Since W is also $\mathcal{S}$-terminal and an $\overline{\mathcal{N}}$-AE-space (as the countable product of $\overline{\mathcal{N}}$-AE-spaces), Theorem 2 implies

Theorem 3. *The space W is a stratified $\overline{\mathcal{N}}$-AE-space. As a consequence, it is an $\mathcal{S}$-AE-space.*

Proof of Theorem 2. Without loss of generality, we may assume that the inequality $\operatorname{diam} X_i \leqslant 1/i$ holds, and the metric d for X is defined by the formula $\mathrm{d}(x, y) = \sup\{\mathrm{d}_i(x_i, y_i)\}$, where d_i is the metric of the space X_i.

Since the class of the $\overline{\mathcal{N}}$-AE-spaces is closed under the countable product operation, we have the inclusion $X \in \overline{\mathcal{N}}$-AE. Hence, there exists an $\overline{\mathcal{N}}$-extension $\widehat{\varphi} : Z \to X$ for the map φ. To prove the theorem, it suffices to find an $\overline{\mathcal{N}}$-map $\psi : Z \to X$ such that $\psi\restriction_A = \widehat{\varphi}\restriction_A$ and $\psi\restriction_{Z\setminus A}$ is an $\mathcal{S}$-map. Since A is closed in Z, it is possible to choose a sequence of neighborhoods $Z = U_0 \Supset U_1 \Supset \cdots$ such that $\bigcap\limits_{i\geqslant 0} U_i = A$. Here the symbol $A \Subset B$ denotes that the embedding A$\subset B$ is *strong*, i.e., the inclusion $\mathrm{Cl}\, A \subset \mathrm{Int}\, B$ holds. For each $i \geqslant 1$ we choose a continuous function $\chi_i : Z \to [0; 1]$ such that $\chi_i^{-1}(0) \supset Z \setminus U_i$, $\chi_i^{-1}(1) \supset U_{i+1}$. It is clear that χ_i is an $\overline{\mathcal{N}}$-map.

Now let us represent the map $\widehat{\varphi} : Z \to X$ as the diagonal product $\prod \widehat{\varphi}_i$, where $\widehat{\varphi}_i : Z \to X_i$ is an $\overline{\mathcal{N}}$-map. Since each X_i is $\mathcal{S}$-terminal, we can take an $\mathcal{S}$-map $e_i : Z \to X_i$. Further, for every $i \geqslant 1$ we consider an $\overline{\mathcal{N}}$-map $h_i : Z \times \{0\} \cup Z \times \{1\} \to X_i$ defined on the closed subset of the space $Z \times [0; 1]$ such that $h_i\restriction_{Z\times\{0\}} = e_i$ and $h_i\restriction_{Z\times\{1\}} = \widehat{\varphi}_i$. Since $X_i \in \overline{\mathcal{N}}$-AE, there exists an $\overline{\mathcal{N}}$-extension $H_i : Z \times [0; 1] \to X_i$ for the map h_i which is an $\overline{\mathcal{N}}$-homotopy between the $\mathcal{S}$-map $e_i : Z \to X_i$ and the $\overline{\mathcal{N}}$-map $\widehat{\varphi}_i : Z \to X_i$.

For every $n \geqslant 1$ we define the map $\xi_n : Z \to X$ by the formula

$$\xi_n = e_1 \times e_2 \times \cdots \times e_{n-1} \times H_n(z, \chi_{n-1}(z)) \times \widehat{\varphi}_{n+1} \times \widehat{\varphi}_{n+2} \times \cdots$$

By Corollary 1, this map is stratified. Next, we set

$$\psi_n(z) = \begin{cases} \xi_n(z), & \text{if } z \in U_{i-1} \setminus U_i,\ i \leqslant n; \\ \widehat{\varphi}(z), & \text{if } z \in U_n. \end{cases}$$

Then each map $\psi_n : Z \to X$ is continuous and stratified on $Z \setminus U_n$.

Since for any numbers $m > n$ the values $\psi_n(z)$ and $\psi_m(z)$ may be distinct starting from the n-th coordinate, we have the estimate $\mathrm{d}(\psi_n(z), \psi_m(z)) < 1/n$ for every $z \in Z$. Hence, the sequence $\{\psi_n\}$ of maps converges uniformly to a continuous map $\psi : Z \to X$ as $n \to \infty$. Moreover, the equalities $\psi\restriction_A = \widehat{\varphi}\restriction_A = \varphi\restriction_A$ are valid. Since almost all of maps ψ_i coincide with ψ_{n+1} outside the neighborhood U_n, the map ψ is stratified on the complement of the set A. $\square$

3. Some properties of the stratified absolute extensors. We will show that any $\mathcal{S}$-A[N]E-space is an $\overline{\mathcal{N}}$-A[N]E-space. To this end, at first we establish the following preliminary fact.

Proposition 2. *Any $\overline{\mathcal{N}}$-space admits a closed $\mathcal{S}$-embedding into an $\mathcal{S}$-AE-space.*

Proof. Consider any $\overline{\mathcal{N}}$-space X. By the Kuratowski–Wojdysławski theorem there exists a closed topological embedding $i : X \to L$ of the space X into a normed linear $\overline{\mathcal{N}}$-AE-space L considering with a trivial filtration. Thus, the embedding $i : X \to L$ is an $\overline{\mathcal{N}}$-map. Furthermore, by Theorem 1, there exists an $\mathcal{S}$-map $f : X \to W$, where W is an $\mathcal{S}$-terminal space. Besides, Theorem 3 implies that we have the inclusion $W \in \mathcal{S}$-AE. Since i is a closed topological embedding and the filtration of the space L is trivial, the map $i \times f : X \to L \times W$ is a closed topological $\mathcal{S}$-embedding into the $\overline{\mathcal{N}}$-space $L \times W$. Finally, it is easy to see that the inclusion $L \times W \in \mathcal{S}$-AE is true. $\square$

Remark. By Theorem 3, the space W is a stratified $\overline{\mathcal{N}}$-AE-space. Since $L \in \overline{\mathcal{N}}$-AE, Corollary 2 implies that the product $L \times W$ is a stratified $\overline{\mathcal{N}}$-AE-space.

Theorem 4. *Any* $\mathcal{S}$-A[N]E-*space* X *is a stratified* $\overline{\mathcal{N}}$-A[N]E-*space* (*and, hence, an* $\overline{\mathcal{N}}$-A[N]E-*space*).

Proof. Let $X \in \mathcal{S}$-AE. To prove the theorem it suffices to show that for any partial $\overline{\mathcal{N}}$-map $Y \hookleftarrow A \xrightarrow{f} X$ one can construct an $\overline{\mathcal{N}}$-extension $\widehat{f} : Y \to X$ to the whole space Y which is stratified on $Y \setminus A$. In view of Proposition 2, there exists a closed $\mathcal{S}$-embedding $j : X \to L \times W$ into an $\mathcal{S}$-AE-space $L \times W$, where W is an $\mathcal{S}$-terminal space from Theorem 1 and L is a normed linear $\overline{\mathcal{N}}$-AE-space with a trivial filtration. According to Remark, the product $L \times W$ is a stratified $\overline{\mathcal{N}}$-AE-space. Hence, the $\overline{\mathcal{N}}$-map $\varphi = j \circ f : A \to L \times W$ can be continuously extended to an $\overline{\mathcal{N}}$-map $\widehat{\varphi} : Y \to L \times W$ which is stratified on $Y \setminus A$. Since the inclusion $X \in \mathcal{S}$-AE holds, there exists an $\mathcal{S}$-retraction $r : L \times W \to X$, i.e., we have $r\restriction_X = \mathrm{Id}$. Then the $\overline{\mathcal{N}}$-map $\widehat{f} = r \circ \widehat{\varphi} : Y \to X$ is the sought-for $\overline{\mathcal{N}}$-extension f which is stratified on $Y \setminus A$.

The case of an $\mathcal{S}$-ANE-space is examined analogously. $\square$

Consider an arbitrary $\overline{\mathcal{N}}$-space $X \in \mathcal{S}$-AE and a partial $\overline{\mathcal{N}}$-map $h : X \times \{0\} \cup X \times \{1\} \to X$ such that $h\restriction_{X\times\{0\}} = \mathrm{Id}$ and $h\restriction_{X\times\{1\}} = x_0$, where $x_0 \in X_{\deg X}$. It follows from Theorem 4 that there exists an $\overline{\mathcal{N}}$-extension $H : X \times [0;1] \to X$ of the map h which is stratified on $X \times (0;1)$. Obviously, the map H is an $\overline{\mathcal{N}}$-homotopy satisfying all conditions for the $\mathcal{S}$-contactibility of the $\overline{\mathcal{N}}$-space X. Thus, we have proved

Theorem 5. *Any* $\mathcal{S}$-AE-*space is* $\mathcal{S}$-*contractible.*

Further, we establish a local variant of this theorem.

Theorem 6. *Any* $\mathcal{S}$-ANE-*space* X *is an* $\mathcal{S}$-LC-*space.*

Proof. To obtain a contradiction we assume that the space X is not locally $\mathcal{S}$-contractible at a point $x_0 \in X$. Since we are given $X \in \mathcal{S}$-ANE, there are points of finite degree in every neighborhood of the point x_0, i.e., x_0 is the limit of a sequence $\{x_k\}_{k\in\mathbb{N}} \in X$ with $\deg x_k < \infty$. We can assume that the inequalities $\deg x_1 \leqslant \deg x_2 \leqslant \deg x_3 \leqslant \cdots$ are valid.

For every $k \in \mathbb{N}$ we consider a neighborhood U_k of the point x_0 with the induced filtration such that the condition $\operatorname{diam} U_k < 2^{-k}$ holds. Setting $U'_k = U_k \cap X_{\deg x_0}$ if $\deg x_0 < \infty$ or $U'_k = U_k$ if $\deg x_0 = \infty$, we can assume that $x_k \in U'_k$. Furthermore, for each $k \in \mathbb{N}$ we define the $\overline{\mathcal{N}}$-space $Z'_k \rightleftharpoons U'_k \times [2^{-2k+1}; 2^{-2k+2}]$ and its closed $\overline{\mathcal{N}}$-subspace $A'_k \rightleftharpoons U'_k \times \{2^{-2k+1}; 2^{-2k+2}\}$.

Since $X \notin \mathcal{S}$-LC at the point x_0, there is a number $\varepsilon_0 > 0$ such that for every neighborhood U of the point x_0 does not exist an $\overline{\mathcal{N}}$-homotopy $H : U \times [0;1] \to X$ satisfying conditions 1)–3) from the definition of the locally $\mathcal{S}$-contractible space. Further, since the inequalities $\operatorname{diam} U'_k < 2^{-k} < \varepsilon_0$ are true for almost all k, the partial $\overline{\mathcal{N}}$-map $Z'_k \hookleftarrow A'_k \xrightarrow{\varphi_k} X$ defined by formulas $\varphi_k\restriction_{U'_k\times\{2^{-2k+1}\}} = \mathrm{Id}$, $\varphi_k\restriction_{U'_k\times\{2^{-2k+2}\}} = x_k$, has no an $\overline{\mathcal{N}}$-extension $\overline{\varphi}_k : Z'_k \to X$ which is stratified on $Z'_k \setminus A'_k$ and satisfies the estimate $\operatorname{diam} \overline{\varphi}_k(Z'_k) < \varepsilon_0$. Notice that we have the equality $\deg x_0 = \underline{\lim} \deg Z'_k$. Indeed, if the condition $\deg x_0 < \infty$ holds, then we get $\underline{\lim} \deg Z'_k = \underline{\lim} \deg U'_k = \deg x_0$. In the case when $\deg x_0 = \infty$ we obtain the following equalities: $\underline{\lim} \deg Z'_k = \underline{\lim} \deg U'_k = \underline{\lim} \deg x_k = \deg x_0 = \infty$.

By Proposition 1, we can consider the one-point complement $Z = \left(\bigsqcup\limits_{k=1}^{\infty} Z'_k\right) \sqcup \{x_0\}$ of the topological sum for the $\overline{\mathcal{N}}$-spaces Z'_k such that the degree of the point x_0 coincides with the degree of x_0 in X. The space Z is a metric $\overline{\mathcal{N}}$-space with the property

$$Z'_k \subset N(x_0; 2^{-k}).$$

Now we define the map $\Phi : A \to X$ whose domain is the closed $\overline{\mathcal{N}}$-subspace $A = \left(\bigsqcup\limits_{k=1}^{\infty} A'_k\right) \sqcup \{x_0\}$ of the $\overline{\mathcal{N}}$-space Z as follows:

$$\Phi(a) = \begin{cases} \varphi_k(a), & \text{if } a \in A'_k,\ k = 1, 2, \dots\ ; \\ x_0, & \text{if } a = x_0. \end{cases}$$

The continuity of the map Φ at the point x_0 follows from the following condition

$$\operatorname{diam} \varphi_k(A'_k) \leqslant \operatorname{diam} U'_k < 2^{-k}.$$

It is clear that Φ is a partial $\overline{\mathcal{N}}$-map taking its values in the $\mathcal{S}$-ANE-space X. It follows from Theorem 4 that there exists a neighboring $\overline{\mathcal{N}}$-extension $\widehat{\Phi} : V \to X$ to a neighborhood V of the set A in Z such that $\widehat{\Phi}\restriction_{V \setminus A}$ is an $\mathcal{S}$-map and the inclusion $\widehat{\Phi}(V) \subset N(x_0; \varepsilon_0)$ holds.

Since $Z'_k \subset N(x_0; 2^{-k})$, starting from a certain number k we have $Z'_k \subset V$It is obvious that the restriction $\overline{\varphi}_k = \widehat{\Phi}\restriction_{Z'_k} : Z'_k \to \widehat{\Phi}(V) \subset X$ is an $\overline{\mathcal{N}}$-extension of the map φ_k to the whole space Z'_k which is stratified on $Z'_k \setminus A$ and satisfies the condition $\operatorname{diam} \overline{\varphi}_k(Z'_k) < \varepsilon_0$. We have obtained the contradiction which proves the theorem. $\square$

A family $\{Y_i\}_{i \in \mathbb{N}}$ of subsets in a metric space Y is said to be *equicontinuously locally extendable* if the following conditions are fulfilled. First, the equality $\bigcup\limits_{i=1}^{\infty} Y_i = Y$ holds. Second, for every point $y \in Y$ and every its neighborhood U there exists a neighborhood $V \subset U$ such that $y \in V$ and for every closed subset A of a metric space Z each continuous map $f : A \to Y_m \cap V$ can be extended to a continuous map $\widehat{f} : Z \to Y_m \cap U$. In this case we write $\{Y_i\}_{i \in \mathbb{N}} \in$ equi-LAE.

Theorem 7. *Let $\{X_n\}_{n \in \overline{\mathbb{N}}}$ be a filtration of an $\mathcal{S}$-*ANE*-space X. Then $\{X^n\}_{n \in \overline{\mathbb{N}}} \in$* equi-LAE.

Proof. Assume, contrary to the assertion of the theorem, that there exist a point $x_0 \in X$ and its neighborhood U satisfying the following condition. For any neighborhood V_k of x_0 such that $V_k \subset U$ and $\operatorname{diam} V_k < 2^{-k}$, where $k \in \mathbb{N}$, there exists a continuous map $Z'_k \hookleftarrow A'_k \xrightarrow{\varphi_k} X^{m_k} \cap V_k$ defined on a closed subspace A'_k of a metric space Z'_k which has no an extension $\overline{\varphi}_k : Z'_k \to X^{m_k} \cap U$. We can assume that the property $V_1 \supset V_2 \supset V_3 \supset \cdots$ holds.

Let us introduce now the filtration in Z'_k by setting $\deg z = m_k$ for $z \in Z'_k$ and consider the topological sum $\bigsqcup\limits_{k=1}^{\infty} Z'_k$ of $\overline{\mathcal{N}}$-spaces. Notice that we have $\deg x_0 \leqslant \lim\limits_{k\to\infty} m_k = \infty$. By Proposition 1, there exist a point s_0 and an $\overline{\mathcal{N}}$-space $Z = \left(\bigsqcup\limits_{k=1}^{\infty} Z'_k\right) \sqcup \{s_0\}$ such that

$$Z'_k \subset N(s_0; 2^{-k}) \ \text{ and } \ \deg s_0 = \deg x_0.$$

Consider the map $\Phi : A \to U$ defined on the closed $\overline{\mathcal{N}}$-subspace $A = \left(\bigsqcup\limits_{k=1}^{\infty} A'_k\right) \sqcup \{s_0\}$ of the $\overline{\mathcal{N}}$-space Z as follows:

$$\Phi(a) = \begin{cases} \varphi_k(a), & \text{if } a \in A'_k,\ k = 1, 2, \dots\ ; \\ x_0, & \text{if } a = s_0. \end{cases}$$

The continuity of the map Φ at the point s_0 follows from the inclusion $Z'_k \subset N(s_0; 2^{-k})$.

For any point $a \in A$ such that $a \in A'_k$, we have $\deg a = m_k$ and $\Phi(a) = \varphi_k(a) \in X^{m_k}$. Moreover, the equalities $\deg \Phi(s_0) = \deg x_0 = \deg s_0$ hold. Consequently, the map Φ is a partial $\mathcal{S}$-map taking its values in the $\mathcal{S}$-ANE-space X. Therefore, there exists a neighborhood $\mathcal{S}$-extension $\widehat{\Phi} : W \to X$ to a neighborhood W of the set A in Z. Since the condition $Z'_k \subset N(s_0; 2^{-k})$ holds, starting from a certain number k we get the inclusion $Z'_k \subset W$. Since the map $\widehat{\Phi}$ is stratified, the restriction $\overline{\varphi}_k = \widehat{\Phi}\restriction_{Z'_k} : Z'_k \to X$ takes its values in the set $X^{m_k} \cap U$. Hence, the map $\overline{\varphi}_k$ is the extension of the map φ_k. This contradiction proves the theorem. $\square$

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